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Modes in Anisotropic Media

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The purpose of this letter is to show that the quasi-geometrical-optics method used in a previous paper[1] to describe the propagation of optical modes and the results obtained there are applicable to anisotropic as well as to isotropic media.

Let us first recall that the ray equations in lossless media are obtained by substituting in the wave equation fields of the form $G(\mathbf{r})\exp[-jkS(\mathbf{r})]$, where $k\equiv 2\pi/\lambda$ denotes the free-space propagation constant and the eikonal $S(\mathbf{r})$ is real. Keeping only the terms with the highest power in k, we obtain an algebraic relation between the cartesian components $p_i\equiv\partial S/\partial x_i$, $i=1$, 2, and $\partial S/\partial z$ of ∇S, which can be written

$$H(p_i,\mathbf{r})+\partial S/\partial z=0. \tag{1}$$

The rays, defined as the lines of flow of power, are perpendicular to the Fresnel surface of wave normals described, at some fixed point $\mathbf{r}$, by the tip of the vector $k\nabla S$. This is expressed by Hamilton's equations for rays $x_i(z)$

$$dx_i/dz = \partial H/\partial p_i, \tag{2a}$$

$$dp_i/dz = -(\partial H/\partial x_i). \tag{2b}$$

These equations, identical with those of classical mechanics, show that the rays of geometrical optics are similar to the trajectories of material particles.[2]

Let us now consider a packet of rays making small angles with one another (though not necessarily with the z axis), and assume that, to one ray, there corresponds only one vector ∇S. For such a packet, we are interested in only a small portion of the Fresnel surface, which can be replaced by a paraboloid. Equation (1) is rewritten

$$f + g_i \frac{\partial S}{\partial x_i} + \tfrac{1}{2} h_{ik} \frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_k} + \frac{\partial S}{\partial z} = 0, \tag{3}$$

where the summation sign over repeated indices is omitted and f, g_1, g_2, h_{11}, h_{22}, and $h_{12}=h_{21}$ are six real functions of $\mathbf{r}$ that can be obtained from the material parameters.

The optical distance from a point x_i', 0 to a point x_i, 1 in a homogeneous medium is easily obtained with the help of Eqs. (2a) and (3). We have, without any further approximation,

$$S(x_i',0;x_i,1) = -f + \tfrac{1}{2}(\mathbf{h}^{-1})_{ik}(x_i' - x_i - g_i)(x_k' - x_k + g_k), \tag{4}$$

where $\mathbf{h}$ is the matrix with elements h_{11}, h_{12}, h_{22}.

In homogeneous media, the point eikonal is therefore a quadratic form in x_i', x_i. When the medium is not homogeneous, it sometimes remains possible to approximate the point eikonal by a form at most quadratic in x_i', x_i (Gauss approximation). In those cases, the relation between x_i', $p_i' \equiv -\partial S/\partial x_i'$ at $z=0$, and x_i, $p_i \equiv \partial S/\partial x_i$ at z, is linear and can be described by a ray matrix as for the case of isotropic media. The expression "ray matrix," however, becomes somewhat inappropriate because the p's are related to the wave normals, and not directly to the rays.

Following the rules of quantum mechanics, we now obtain the paraxial wave equation by ordering Eq. (3), and replacing $\partial S/\partial x_i$ and $\partial S/\partial z$ by the operators $jk^{-1}\partial/\partial x_i$ and $jk^{-1}\partial/\partial z$, respectively.[3] We obtain

$$\left[f + \tfrac{1}{2} jk^{-1}\left(g_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} g_i \right) + \tfrac{1}{2}(jk^{-1})^2 \frac{\partial}{\partial x_i} h_{ik} \frac{\partial}{\partial x_k} + jk^{-1}\frac{\partial}{\partial z} \right]\psi = 0, \tag{5}$$

where ψ is a scalar field function such that $\psi\psi^* ds$ represents the power flowing through a small area ds in the plane z.

Let us evaluate the geometrical-optics field created by a point source and find under what conditions this field is an exact Green function of the paraxial wave equation, Eq. (5).

Consider an incident field ψ' at $\mathbf{r}'$ uniform over an area ds' whose dimensions are of the order of a few wavelengths, and zero elsewhere. The area covered in p space by the components of the wave normals in this beam at z' is of the order of λ^2/ds', because the phase shift cannot exceed π within the area ds'.

Let us consider next a cone of rays originating from $\mathbf{r}'$ whose end points cover an area δs at z. The area covered in p space by the wave normals at z' is easily found to be $|\partial^2 S(\mathbf{r}';\mathbf{r})/\partial x_i' \partial x_k|\delta s$, where the vertical bars denote a determinant.

If we now observe that the power flowing through two homocentric cones of rays is in direct proportion to their solid angles, and also in direct proportion to the area covered in p space by the wave normals, because small increments are considered, power conservation entails that

$$\psi\psi^* \delta s = \psi'\psi'^* ds' \frac{|\partial^2 S/\partial x_i' \partial x_k|\delta s}{\lambda^2/ds'}, \tag{6}$$

to within a numerical factor that can be shown to be unity.

The phase of ψ, on the other hand, is $\pi/2 - kS$, mod. π, if the anomalous phase shifts in the neighborhood of $\mathbf{r}'$ and subsequent foci is taken into account. We thus obtain for the geometrical-optics field created by a point source at $\mathbf{r}'$ the Van Vleck propagator[3]

$$\psi(\mathbf{r};\mathbf{r}') = \pm j\lambda^{-1}|\partial^2 S/\partial x_i' \partial x_k|^{\frac{1}{2}} \exp(-jkS), \tag{7}$$

as for the case of isotropic media.

Substituting the right-hand side of Eq. (7) in Eq. (5), we find that the terms proportional to $(jk^{-1})^0$ and $(jk^{-1})^1$ vanish, in agreement with the correspondence principle.[3] The last term, proportional to $(jk^{-1})^2$,

$$\pm \tfrac{1}{2} j\lambda^{-1} \exp(-jkS)(jk^{-1})^2 \frac{\partial}{\partial x_i} h_{ik} \frac{\partial}{\partial x_k} |\partial^2 S/\partial x_i' \partial x_k|^{\frac{1}{2}}, \tag{8}$$

clearly vanishes also when S is at most quadratic in x_i, x_i', for every z, z', i.e., within the approximation of Gauss. In that case, the right-hand side of Eq. (7) is the exact Green function of the paraxial wave equation.

This expression, Eq. (7), is applicable to optical systems with nonuniform losses by introduction of complex point eikonals.[1] The multipole expansion of the field created by a point source through an arbitrary lossy and misaligned optical system (called in Ref. 1 a "mode-generating system") readily gives the modes of propagation, expressible as products of Gauss functions and Hermite polynomials in two complex variables. Because we need consider here misaligned mode-generating systems, the wave fronts of modes associated with anisotropic media are generally tilted, possibly at a large angle, with respect to the beam axes.

A difficulty arises in optics from the fact that the Fresnel surface exhibits two shells, corresponding to two orthogonal eigenstates of polarization.[2] Thus, to a given ray there correspond two waves and not just one as assumed before. However, if the beat wavelength between these two waves is small compared with the scale of the inhomogeneities, the waves remain uncoupled and the results given in this letter are applicable to each eigenstate of polarization independently. The present theory is generally not applicable if there are abrupt spatial changes in the medium.

[1] J. A. Arnaud, J. Opt. Soc. Am. **61**, 751 (1971). [Note that a subscript s is missing at the first 𝔐 after Eq. (31).] It recently came to our attention that the representation of fundamental gaussian beams by complex rays that we introduced in other publications [e.g., Appl. Opt. **8**, 189 (1969), footnote on p. 190] had been previously considered by G. Deschamps (U.R.S.I. Symposium, Stresa, 1968).

[2] M. Kline and I. W. Kay, *Electromagnetic Theory and Geometrical Optics* (Wiley–Interscience, New York, 1965), Ch. III.

[3] J. H. Van Vleck, Proc. Natl. Acad. Sci. (U. S.) **14**, 178 (1928).