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Mode Coupling in First-Order Optics

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A simple procedure is described to obtain the modes of propagation in square-law lens-like media. This procedure consists of evaluating the geometrical-optics field created by a point source at the input plane of an optical system (called mode-generating system) with nonuniform losses. An expansion of the field in power series of the coordinates of the point source gives the modes of propagation. In the case of optical resonators, the mode-generating system is described by the modal matrix of the resonator round-trip ray matrix. This representation of modes by point sources allows the coupling factor between two modes with different parameters (beam radii, wave-front curvatures, and axes) to be evaluated without integration. Only matrix algebra is used. In the general three-dimensional case, the coupling factor is expressed as a product of Gauss functions and Hermite polynomials in four complex variables. The quantities introduced are generalized ray invariants.

INDEX HEADINGS: Resonant modes; Geometrical optics; Optical systems; Lasers.

It is well known that, within the first-order approximation, the modes of propagation of optical beams in two-dimensional optical systems are Hermite–Gauss functions.[1] In the more general case of three-dimensional optical systems that lack meridional planes of symmetry, it is necessary to introduce Hermite polynomials in two complex variables.[2] A simple procedure is described in this paper to obtain these generalized modes and to calculate the coupling factor between modes corresponding to different parameters (beam radii, wave-front curvatures, and axes). The latter result is useful in evaluating, for example, the power transfer between a laser source and a resonator, or the response of heterodyne optical receivers. The expression obtained generalizes previous results.

In the first section, a few results of paraxial geometrical optics are recalled.

THE GEOMETRICAL-OPTICS FIELD

Consider a lossless optical system with an input reference plane (x) and an output reference plane (x'). The optical distance between a point $\mathbf{x}$ at the input plane and a point $\mathbf{x}'$ at the output plane is extremum for a certain path called a ray; it is assumed that this path is unique. The optical length of this ray is called the point characteristic of the optical system and is denoted $V(\mathbf{x},\mathbf{x}')$.

The point where a ray intersects the input plane is defined by a position vector $\mathbf{q}$, represented by a column matrix in a x_1x_2 rectangular coordinate system. The projection on the reference plane of a vector directed along the ray having a length equal to the refractive index n is called the direction vector of the ray and is denoted $\mathbf{p}$. We similarly define the position and direction vectors $\mathbf{q}'$, $\mathbf{p}'$ of the ray at the output plane.

As is well known, $\mathbf{p}$ and $\mathbf{p}'$ can be obtained from the point characteristic $V(\mathbf{x},\mathbf{x}')$ by differentiation[3]

$$\mathbf{p}=-\nabla_{\mathbf{x}}V(\mathbf{x},\mathbf{x}'), \tag{1a}$$

$$\mathbf{p}'=\nabla_{\mathbf{x}'}V(\mathbf{x},\mathbf{x}'), \tag{1b}$$

at $\mathbf{x}=\mathbf{q}$, $\mathbf{x}'=\mathbf{q}'$. In Eqs. (1a) and (1b), $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{x}'}$ denote gradient operators in the (x) and (x') planes, respectively.

Let us now consider a point source located at a point $\mathbf{x}$ of the input plane. The geometrical-optics scalar field created by this point source at the output plane is, within the paraxial approximation,[4,5]

$$E(\mathbf{x}';\mathbf{x})=\pm j\lambda^{-1}|\partial^2V/\partial x_i\partial x_j'|^{\frac{1}{2}}\exp(-jkV), \tag{2}$$

where $k\equiv 2\pi/\lambda$ denotes the free-space propagation constant and the vertical bars denote a determinant. For simplicity, we assume that the refractive index is unity on axis. The physical significance of the exponential term in Eq. (2) is obvious. It expresses the phase shift resulting from the optical length V. The term $|\partial^2V/\partial x_i\partial x_j'|^{\frac{1}{2}}$ is obtained by recognizing that the power flowing through a small area at the output plane is equal to the power flowing in the corresponding cone of rays leaving the point source, and using Eq. (1). The factor j can be viewed as an anomalous phase shift at the point source, similar to the one observed at focal points of ray pencils. The sign ambiguity in Eq. (2) can be lifted only if the number of focal lines existing on the ray pencil that originates from the point source is known. Knowledge of the point characteristic is therefore not quite sufficient to determine the field at the output plane. This turns out to be, however, a restriction of minor importance. Note also that for beams propagating from the output to the input plane k in Eq. (2) should be changed to $-k$.

Let us now assume that the point characteristic can be approximated by a quadratic form

$$V(\mathbf{x},\mathbf{x}')=L+\tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{U}\mathbf{x}+\tilde{\mathbf{x}}\mathbf{V}\mathbf{x}'+\tfrac{1}{2}\tilde{\mathbf{x}}'\mathbf{W}\mathbf{x}', \tag{3}$$

where L denotes the optical length of the optical axis. $\mathbf{U}$ and $\mathbf{W}$ denote two 2×2 symmetric matrices, $\mathbf{V}$ a 2×2 matrix, and $\sim$ indicates matrix transposition.

Introducing Eq. (3) in Eq. (1), we get the paraxial-ray equations

$$-\mathbf{p}=\mathbf{U}\mathbf{q}+\mathbf{V}\mathbf{q}' \tag{4a}$$

$$\mathbf{p}'=\tilde{\mathbf{V}}\mathbf{q}+\mathbf{W}\mathbf{q}'. \tag{4b}$$

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The field of a point source is, from Eqs. (3) and (2)

$$E(\mathbf{x}';\mathbf{x})=\pm j\lambda^{-1}|\mathbf{V}|^{\frac{1}{2}} \times\exp[-jk(L+\tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{U}\mathbf{x}+\tilde{\mathbf{x}}\mathbf{V}\mathbf{x}'+\tfrac{1}{2}\tilde{\mathbf{x}}'\mathbf{W}\mathbf{x}')]. \quad (5)$$

This expression, Eq. (5), can be used as a Green function to obtain the transformation of an arbitrary incident field $E(\mathbf{x})$. The field at the output plane is

$$E'(\mathbf{x}')=\int\int_{-\infty}^{+\infty} E(\mathbf{x}';\mathbf{x})E(\mathbf{x})dx_1dx_2. \quad (6)$$

Notice, incidentally, that if the optical system is empty, Eq. (6) reduces to the Fresnel approximation of the Kirchhoff integral.

If we want to know the field transformation through a sequence of two, or more, optical systems, there is no need for applying Eq. (6) repeatedly. It suffices to evaluate the total point characteristic by adding the point characteristics of each individual optical system and eliminating the intermediate variables, and apply Eq. (6) only once. The proof that this procedure is valid can be found in Ref. 6 or 2.

This result allows us to apply Eq. (5) to systems with nonuniform losses, provided that these losses can be approximated by quadratic laws (in decibels). Consider, for instance, an apodized aperture introducing on incident fields an attenuation of the form $\exp[-(x/w)^2]$ where w denotes some effective radius. This aperture is formally equivalent to a thin lens with an imaginary focal length $-jkw^2/2$. The point characteristic of an arbitrary sequence of lossless systems and apodized apertures can therefore be obtained by applying formally the rule of point-characteristic addition. This point characteristic is, in general, a complex function of x and x'.

The subsequent calculations could be based only on the consideration of point characteristics. It is, however, convenient to introduce also the ray matrices which relate the output quantities $\mathbf{q}'$, $\mathbf{p}'$ to the input quantities $\mathbf{q}$, $\mathbf{p}$

$$\begin{bmatrix}\mathbf{q}'\\ \mathbf{p}'\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix}\equiv\mathfrak{M}\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix}. \quad (7)$$

By comparing Eqs. (4) and (7), we readily obtain

$$\mathbf{U}=\mathbf{B}^{-1}\mathbf{A}, \quad (8a)$$

$$\mathbf{V}=-\mathbf{B}^{-1}, \quad (8b)$$

$$\tilde{\mathbf{V}}=\mathbf{C}-\mathbf{D}\mathbf{B}^{-1}\mathbf{A}, \quad (8c)$$

$$\mathbf{W}=\mathbf{D}\mathbf{B}^{-1}. \quad (8d)$$

Luneburg's relations[3] are obtained from Eq. (8) if we remember that $\mathbf{U}$ and $\mathbf{W}$ are symmetrical. We have in particular

$$\tilde{\mathbf{B}}\mathbf{D}-\tilde{\mathbf{D}}\mathbf{B}=0, \quad (9a)$$

$$\tilde{\mathbf{C}}\mathbf{A}-\tilde{\mathbf{A}}\mathbf{C}=0, \quad (9b)$$

$$\tilde{\mathbf{D}}\mathbf{A}-\tilde{\mathbf{B}}\mathbf{C}=1. \quad (9c)$$

These relations show that the inverse of $\mathfrak{M}$ is

$$\mathfrak{M}^{-1}=\begin{bmatrix}\tilde{\mathbf{D}} & -\tilde{\mathbf{B}}\\ -\tilde{\mathbf{C}} & \tilde{\mathbf{A}}\end{bmatrix}. \quad (10)$$

We will find it convenient to introduce also matricial rays $\mathbf{Q}$, $\mathbf{P}$ defined by

$$\mathbf{Q}\equiv[\mathbf{q}_1\mathbf{q}_2], \quad (11a)$$

$$\mathbf{P}\equiv[\mathbf{p}_1\mathbf{p}_2], \quad (11b)$$

where $\mathbf{q}_1$, $\mathbf{p}_1$ and $\mathbf{q}_2$, $\mathbf{p}_2$ denote any two rays satisfying Eq. (7). Clearly, we have

$$\begin{bmatrix}\mathbf{Q}'\\ \mathbf{P}'\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix}. \quad (12)$$

Let us now consider two rays with position and direction vectors $\bar{\mathbf{q}}_\alpha$, $\bar{\mathbf{p}}_\alpha$ and $\bar{\mathbf{q}}_\beta$, $\bar{\mathbf{p}}_\beta$, respectively. The scalar quantity

$$\tilde{\bar{\mathbf{q}}}_\alpha\bar{\mathbf{p}}_\beta-\tilde{\bar{\mathbf{p}}}_\alpha\bar{\mathbf{q}}_\beta \quad (13)$$

is known as the Lagrange invariant[3]; it assumes the same value at the output and input plane of an optical system, as we easily verify with the help of Eq. (4). For later convenience, we denote this invariant

$$\tilde{\bar{\mathbf{q}}}_\alpha\mathbf{p}_\beta-\tilde{\bar{\mathbf{p}}}_\alpha\bar{\mathbf{q}}_\beta\equiv 4jk^{-1}(\bar{\alpha};\bar{\beta}). \quad (14)$$

We also define a matricial ray invariant

$$\tilde{\mathbf{Q}}_\alpha\mathbf{P}_\beta-\tilde{\mathbf{P}}_\alpha\mathbf{Q}_\beta\equiv 4jk^{-1}(\alpha;\beta). \quad (15)$$

Up to now we have implicitly assumed that the optical system is aligned, i.e., that the point characteristic does not contain linear terms in $\mathbf{x}$, $\mathbf{x}'$. In the more general case of misaligned optical systems, 5×5 ray matrices have to be introduced. We have, in that case, instead of Eq. (7)

$$\begin{bmatrix}\mathbf{q}'\\ \mathbf{p}'\\ 1\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{a}\\ \mathbf{C} & \mathbf{D} & \mathbf{c}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\\ 1\end{bmatrix}\equiv\mathfrak{M}\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\\ 1\end{bmatrix}, \quad (16a)$$

where $\mathbf{a}$ and $\mathbf{c}$ denote vectors. The inverse of $\mathfrak{M}$ is

$$\mathfrak{M}^{-1}=\begin{bmatrix}\tilde{\mathbf{D}} & -\tilde{\mathbf{B}} & \tilde{\mathbf{B}}\mathbf{c}-\tilde{\mathbf{D}}\mathbf{a}\\ -\tilde{\mathbf{C}} & \tilde{\mathbf{A}} & \tilde{\mathbf{C}}\mathbf{a}-\tilde{\mathbf{A}}\mathbf{c}\\ 0 & 0 & 1\end{bmatrix}. \quad (16b)$$

The point characteristic now contains a linear term $\tilde{\mathbf{u}}\mathbf{x}+\tilde{\mathbf{u}}'\mathbf{x}'$, where

$$\mathbf{u}=\mathbf{B}^{-1}\mathbf{a} \quad (17a)$$

$$\mathbf{u}'=\mathbf{c}-\mathbf{D}\mathbf{B}^{-1}\mathbf{a}. \quad (17b)$$

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The expression, Eq. (13), is no longer invariant. To define a ray invariant, we must introduce an auxiliary ray $\gamma(\bar{\mathbf{q}}_\gamma,\bar{\mathbf{p}}_\gamma)$ and set

$$(\tilde{\bar{\mathbf{q}}}_\alpha-\tilde{\bar{\mathbf{q}}}_\gamma)(\bar{\mathbf{p}}_\beta-\bar{\mathbf{p}}_\gamma)-(\tilde{\bar{\mathbf{p}}}_\alpha-\tilde{\bar{\mathbf{p}}}_\gamma)(\bar{\mathbf{q}}_\beta-\bar{\mathbf{q}}_\gamma) \equiv 4jk^{-1}(\bar{\alpha}-\bar{\gamma};\bar{\beta}-\bar{\gamma}). \quad (18)$$

We will also make use of a vectorial ray invariant

$$\tilde{\mathbf{Q}}_\alpha(\bar{\mathbf{p}}_\beta-\bar{\mathbf{p}}_\gamma)-\tilde{\mathbf{P}}_\alpha(\bar{\mathbf{q}}_\beta-\bar{\mathbf{q}}_\gamma)\equiv 4jk^{-1}(\alpha;\bar{\beta}-\bar{\gamma}), \quad (19)$$

where it is assumed that $\mathbf{Q}_\alpha$, $\mathbf{P}_\alpha$ obey the ray equations with no misalignment terms.

The transition between ray optics and wave optics is now to be made simply by allowing the ray components to assume complex values. This procedure is discussed in the next section.

MODES OF PROPAGATION

Let us consider an optical system ($\mathfrak{G}$) described by a ray matrix

$$\mathfrak{M}\equiv\tfrac{1}{2}\begin{bmatrix}jk\mathbf{Q}^\ddagger & \mathbf{Q}\\ jk\mathbf{P}^\ddagger & \mathbf{P}\end{bmatrix}. \quad (20)$$

Because $\mathfrak{M}$ is a ray matrix, it must satisfy the Luneburg rule of inversion, Eq. (10). The 2×2 submatrices $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{Q}^\ddagger$, $\mathbf{P}^\ddagger$ are otherwise arbitrary and may assume complex values. According to Eqs. (9a), (9b), and (9c), we have

$$\mathbf{Q}\tilde{\mathbf{P}}-\mathbf{P}\tilde{\mathbf{Q}}=0, \quad (21a)$$

$$\tilde{\mathbf{Q}}^\ddagger\mathbf{P}^\ddagger-\tilde{\mathbf{P}}^\ddagger\mathbf{Q}^\ddagger=0, \quad (21b)$$

$$\tilde{\mathbf{Q}}\mathbf{P}^\ddagger-\tilde{\mathbf{P}}\mathbf{Q}^\ddagger=4jk^{-1}1. \quad (21c)$$

These relations show, in particular, that $\mathbf{PQ}^{-1}$ and $\mathbf{Q}^{-1}\mathbf{Q}^\ddagger$ are symmetrical matrices. Note that no special significance is attached to the sign $\ddagger$. However, if we are interested only in lossless optical systems, $\ddagger$ can be replaced by a star indicating complex-conjugate values. We call the optical system ($\mathfrak{G}$) a mode-generating system for a reason that will now become clear.

Let (y) denote the input plane of this system, and (x) the output plane. The field created at $\mathbf{x}$ by a point source at $\mathbf{y}$ is obtained from Eqs. (5), (8), and (20). We have

$$E(\mathbf{x};\mathbf{y})=\pm(\pi/2)^{-\frac{1}{2}}|\mathbf{Q}|^{-\frac{1}{2}}\exp\left(-j\frac{k}{2}\tilde{\mathbf{x}}\mathbf{PQ}^{-1}\mathbf{x}\right) \times\exp(2jk\tilde{\mathbf{y}}\mathbf{Q}^{-1}\mathbf{x}-\tfrac{1}{2}jk\tilde{\mathbf{y}}\mathbf{Q}^{-1}\mathbf{Q}^\ddagger jk\mathbf{y}). \quad (22)$$

A constant factor has been introduced in Eq. (22) for later convenience. The geometrical-optics phase shift is omitted for brevity.

Let us first assume that the source is located on axis ($\mathbf{y}=0$). The field

$$E(\mathbf{x};0)=\pm(\pi/2)^{-\frac{1}{2}}|\mathbf{Q}|^{-\frac{1}{2}}\exp\left(-j\frac{k}{2}\tilde{\mathbf{x}}\mathbf{PQ}^{-1}\mathbf{x}\right) \quad (23)$$

represents a gaussian field, provided that the imaginary part of the complex symmetrical matrix $\mathbf{M}\equiv\mathbf{PQ}^{-1}$ is definite negative. When this condition is satisfied, the beam represented by $E(\mathbf{x};0)$ carries a finite power at plane (x) because, in that case, the field amplitude decreases exponentially as $x_1^2+x_2^2$ tends to infinity. In the case of lossless systems the field amplitude assumes the form

$$\exp(-\tilde{\mathbf{x}}\tilde{\mathbf{Q}}^{-1}\mathbf{Q}^{*-1}\mathbf{x}),$$

which clearly describes a gaussian irradiance pattern. This expression is obtained with the help of Eq. (21c), where $\ddagger$ is replaced by a star.

Consider now off-set point sources ($\mathbf{y}\neq0$). The second exponential term in Eq. (22) is the generating function for Hermite polynomials in two variables[7]; $E(\mathbf{x};\mathbf{y})$ can be expanded in power series of jky_1 and jky_2. We have

$$E(\mathbf{x};\mathbf{y})=\sum_{m_1,m_2=0}^{\infty}(m_1!m_2!)^{-1} \times(jky_1)^{m_1}(jky_2)^{m_2}E_{m_1m_2}(\mathbf{x}), \quad (24)$$

where

$$E_{m_1m_2}(\mathbf{x})=\pm(\pi/2)^{-\frac{1}{2}}(m_1!m_2!)^{-\frac{1}{2}}|\mathbf{Q}|^{-\frac{1}{2}} \times\exp\left(-j\frac{k}{2}\tilde{\mathbf{x}}\mathbf{PQ}^{-1}\mathbf{x}\right)He_{m_1m_2}(2\mathbf{Q}^{\ddagger-1}\mathbf{x};\ \mathbf{Q}^{-1}\mathbf{Q}^\ddagger). \quad (25)$$

Explicit expressions for Hermite polynomials $He_{m_1m_2}(\mathbf{x};\mathbf{v})$ are given in the Appendix.

To prove that the field $E_{m_1m_2}(\mathbf{x})$ actually represents a mode of propagation, let us assume that the mode-generating system ($\mathfrak{G}$) is followed by an optical system ($\mathfrak{S}$) described by a ray matrix $\mathfrak{M}_s$ with an input plane (x) and an output plane (x'), as shown in Fig. 1. Instead of evaluating the transformation of the field $E_{m_1m_2}(\mathbf{x})$ by the integral transformation Eq. (6), it is simpler to calculate the total ray matrix corresponding to the two optical systems ($\mathfrak{G}$) and ($\mathfrak{S}$) in sequence,

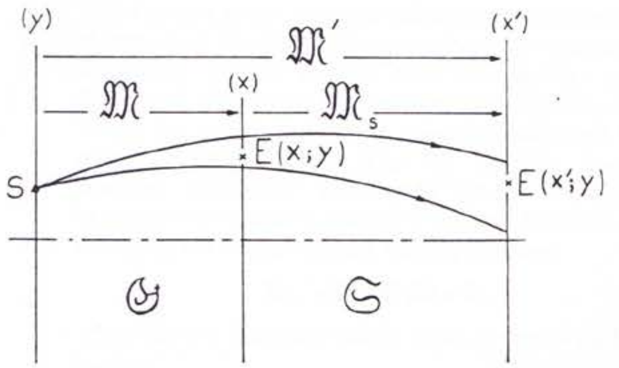

FIG. 1. In this figure ($\mathfrak{S}$) denotes an optical system having a ray matrix $\mathfrak{M}_s$. The modes of propagation in this system are obtained by considering the geometrical-optics field created by a point source S through a mode-generating system ($\mathfrak{G}$) whose ray matrix $\mathfrak{M}$ is generally complex.

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and apply Eq. (5). Let us therefore evaluate the ray matrix

$$\mathfrak{M}'=\mathfrak{M}_s\mathfrak{M}. \tag{26}$$

Setting

$$\mathfrak{M}_s=\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{27}$$

and

$$\mathfrak{M}'=\tfrac{1}{2}\begin{bmatrix} jkQ^{\ddagger\prime} & Q' \\ jkP^{\ddagger\prime} & P' \end{bmatrix}, \tag{28}$$

in Eq. (26), we get

$$\begin{bmatrix} Q' \\ P' \end{bmatrix}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} Q \\ P \end{bmatrix}, \tag{29a}$$

and

$$\begin{bmatrix} Q^{\ddagger\prime} \\ P^{\ddagger\prime} \end{bmatrix}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} Q^{\ddagger} \\ P^{\ddagger} \end{bmatrix}. \tag{29b}$$

Comparing these expressions with Eq. (12), we see that the complex matrices $\mathbf{Q}$, $\mathbf{P}$ on the one hand, and $\mathbf{Q}^{\ddagger}$, $\mathbf{P}^{\ddagger}$ on the other hand, are transformed in the same way as rays. The transformation of the wave-front complex curvature $\mathbf{M}\equiv\mathbf{P}\mathbf{Q}^{-1}$ of the fundamental mode is readily obtained from Eq. (29a). We have

$$\mathbf{M}'\equiv\mathbf{P}'\mathbf{Q}'^{-1}=(\mathbf{C}+\mathbf{D}\mathbf{M})(\mathbf{A}+\mathbf{B}\mathbf{M})^{-1}. \tag{30}$$

To conclude this section, let us briefly indicate how the modes of propagation just defined are related to the modes of resonance of optical resonators. Let a

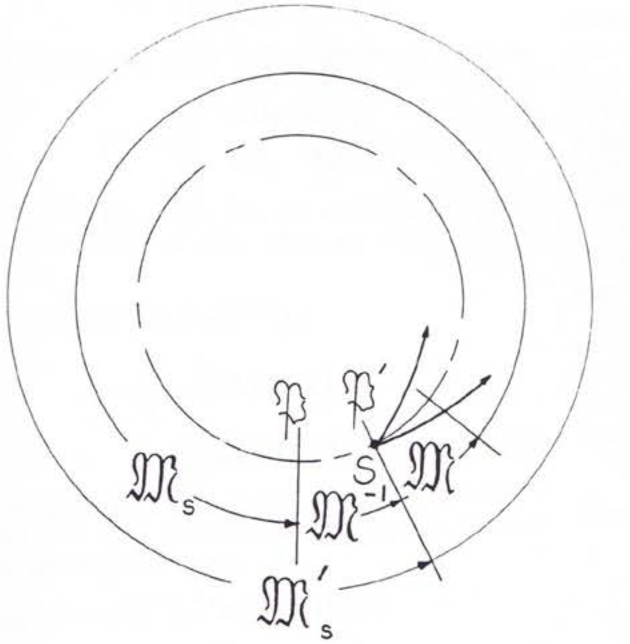

FIG. 2. This figure represents schematically a ring-type resonator having a round-trip ray matrix $\mathfrak{M}_s$, defined from a plane $\mathfrak{P}$. With respect to the plane $\mathfrak{P}'$, the round-trip ray matrix is the diagonal matrix $\mathfrak{M}^{-1}\mathfrak{M}_s\mathfrak{M}$, where $\mathfrak{M}$ is the modal matrix of the resonator. The modes of resonance can be viewed as resulting from a multipole expansion of a point source S at $\mathfrak{P}'$.

resonator be defined by its round-trip ray matrix $\mathfrak{M}_s$, defined from some arbitrary reference plane $\mathfrak{P}$ (see Fig. 2).

It is easy to show, using Eq. (4), that the eigenvalues of a ray matrix can be written λ_1^{-1}, λ_2^{-1}, λ_1, and λ_2. If the corresponding eigenrays (eigenvectors) are denoted $j(k/2)\mathfrak{R}_1^{\ddagger}$, $j(k/2)\mathfrak{R}_2^{\ddagger}$, $\frac{1}{2}\mathfrak{R}_1$, and $\frac{1}{2}\mathfrak{R}_2$, respectively, the modal matrix is

$$\mathfrak{M}\equiv\tfrac{1}{2}[jk\mathfrak{R}_1^{\ddagger}jk\mathfrak{R}_2^{\ddagger}\mathfrak{R}_1\mathfrak{R}_2]. \tag{31}$$

The ordering of these eigenrays is defined by the condition that the mode power is finite. In the case of stable lossless resonators, such an ordering always exists and is unique.[2]

Using the orthogonality property of eigenvectors of $\mathfrak{M}$ and of its transpose corresponding to distinct eigenvalues, it can be shown that the modal matrix $\mathfrak{M}$, Eq. (31), is itself a ray matrix, i.e., that it satisfies the rule of inversion [Eq. (10)]. With the notation of Eq. (11), $\mathfrak{M}$ can be rewritten in the form [Eq. (20)] of the ray matrix of a mode-generating system. The modes of resonance of an optical resonator are consequently obtained by taking for $\mathfrak{M}$ in Eq. (25) the modal matrix of the resonator.

After a round trip, the eigenrays become, by definition, $j(k/2)\lambda_1^{-1}\mathfrak{R}_1^{\ddagger}$, $j(k/2)\lambda_2^{-1}\mathfrak{R}_2^{\ddagger}$, $\frac{1}{2}\lambda_1\mathfrak{R}_1$, and $\frac{1}{2}\lambda_2\mathfrak{R}_2$, respectively. It is not difficult to see that under this transformation the mode field $E_{m_1m_2}(\mathbf{x})$ given in Eq. (25) reproduces itself, except for a constant factor $\pm\exp(-jkL)\lambda_1^{m_1+\frac{1}{2}}\lambda_2^{m_2+\frac{1}{2}}$.

The self-consistency of the field in the resonator therefore requires that

$$\pm\exp(-jkL)\lambda_1^{m_1+\frac{1}{2}}\lambda_2^{m_2+\frac{1}{2}}=\exp(2jl\pi), \tag{32}$$

where l is an integer called the axial-mode number. Equation (32) gives the resonant frequencies and the losses of the resonator. Because $kL\gg1$, changing the sign from + to − in Eq. (32) simply offsets the resonant frequencies of all the modes by a small amount. The sign ambiguity in Eq. (32) is therefore relatively unimportant.

Let us now give a physical interpretation of the modes of resonance of optical resonators based on the previous discussion. Let us suppose that two optical systems $(\mathfrak{G}^{-1})$ and $(\mathfrak{G})$ with ray matrices $\mathfrak{M}^{-1}$ and $\mathfrak{M}$, respectively, are introduced in the resonator, adjacent to the reference plane $\mathfrak{P}$ (see Fig. 2). Nothing has been changed, physically, because $\mathfrak{M}^{-1}\mathfrak{M}=1$. Taking now as a reference plane the plane $\mathfrak{P}'$ located between $(\mathfrak{G}^{-1})$ and $(\mathfrak{G})$, the round-trip ray matrix becomes

$$\mathfrak{M}_s'=\mathfrak{M}^{-1}\mathfrak{M}_s\mathfrak{M}=\mathfrak{D}, \tag{33}$$

where $\mathfrak{D}$ is a diagonal matrix with elements λ_1^{-1}, λ_2^{-1}, λ_1, λ_2.

The modes of resonance can therefore be viewed as resulting from the multipole expansion of a point source located at a certain plane $\mathfrak{P}'$ in the resonator. It should be kept in mind, however, that $\mathfrak{M}$ and $\mathfrak{M}^{-1}$ represent

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optical systems with nonuniform losses and nonuniform gains, respectively. The plane with respect to which the round-trip ray matrix is diagonal does not necessarily exist in a particular resonator.

COUPLING BETWEEN COAXIAL MODES

We have obtained in the previous section a general expression for the field of the modes of propagation in square-law media. This expression depends on a complex matrix $\mathfrak{M}$ that plays the role of a parameter. We wish now to evaluate the coupling factor between two modes with different mode numbers and different matrices $\mathfrak{M}$, having the same axis.

Let $E_\alpha(\mathbf{x},z)$ and $E_\beta^{\ddagger}(\mathbf{x},z)$ denote the fields of two optical beams that propagate, respectively, in the $+z$ and $-z$ directions, z being the axial coordinate.

The coupling factor between these two beams is defined by the integral

$$C_{\alpha\beta}=\int\int_{-\infty}^{+\infty}E_\alpha(\mathbf{x},z)E_\beta^{\ddagger}(\mathbf{x},z)dx_1dx_2. \tag{34}$$

It is easy to show[8] that $C_{\alpha\beta}$ is independent of z. When E_α and $E_\beta^{\ddagger}$ represent two modes of propagation, described by complex rays, we therefore expect $C_{\alpha\beta}$ to be expressible in terms of ray invariants. We shall see that this is indeed the case.

Let us take for $E_\alpha(\mathbf{x})$ the mode m_1m_2 generated by a ray matrix

$$\mathfrak{M}_\alpha=\tfrac{1}{2}\begin{bmatrix} jk\mathbf{Q}_\alpha^{\ddagger} & \mathbf{Q}_\alpha \\ jk\mathbf{P}_\alpha^{\ddagger} & \mathbf{P}_\alpha \end{bmatrix}, \tag{35}$$

and for $E_\beta^{\ddagger}(\mathbf{x})$ the mode $m_1'm_2'$ generated by a ray matrix

$$\mathfrak{M}_\beta^{\ddagger}=\tfrac{1}{2}\begin{bmatrix} -jk\mathbf{Q}_\beta & \mathbf{Q}_\beta^{\ddagger} \\ -jk\mathbf{P}_\beta & \mathbf{P}_\beta^{\ddagger} \end{bmatrix}. \tag{36}$$

Substituting Eq. (35) in Eqs. (8) and (5) and expanding in power series of jky_1, jky_2, we obtain for the mode field

$$E_{\alpha m_1m_2}(\mathbf{x})=(\pi/2)^{-\frac{1}{2}}(m_1!m_2!)^{-\frac{1}{2}}|\mathbf{Q}_\alpha|^{-\frac{1}{2}}\times\exp\left(-j\frac{k}{2}\tilde{\mathbf{x}}\mathbf{P}_\alpha\mathbf{Q}_\alpha^{-1}\mathbf{x}\right)\times He_{m_1m_2}(2\mathbf{Q}_\alpha^{\ddagger-1}\mathbf{x};\,\mathbf{Q}_\alpha^{-1}\mathbf{Q}_\alpha^{\ddagger}). \tag{37}$$

Substituting Eq. (36) in Eqs. (8) and (5) (with k changed into $-k$), and expanding in power series of $-jky_1^{\ddagger}$ and $-jky_2^{\ddagger}$, we get

$$E_{\beta m_1'm_2'}^{\ddagger}(\mathbf{x})=(\pi/2)^{-\frac{1}{2}}(m_1'!m_2'!)^{-\frac{1}{2}}|\mathbf{Q}_\beta^{\ddagger}|^{-\frac{1}{2}}\times\exp\left(j\frac{k}{2}\tilde{\mathbf{x}}\mathbf{P}_\beta^{\ddagger}\mathbf{Q}_\beta^{\ddagger-1}\mathbf{x}\right)\times He_{m_1'm_2'}(2\mathbf{Q}_\beta^{-1}\mathbf{x};\,\mathbf{Q}_\beta^{\ddagger-1}\mathbf{Q}_\beta). \tag{38}$$

Note that the fields E_α and $E_\beta^{\ddagger}$ have been normalized.

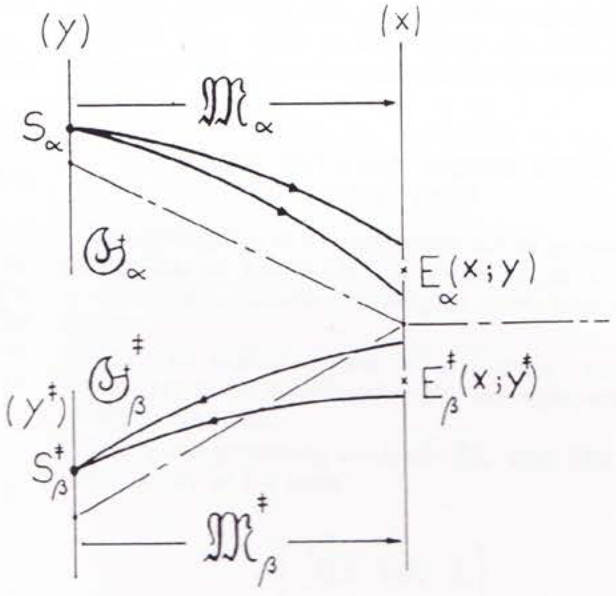

FIG. 3. This figure indicates how to evaluate the coupling factor at plane (x) between a mode generated by a point source S_α through the mode-generating system $(\mathfrak{G}_\alpha)$, and a mode generated by a point sink $S_\beta^{\ddagger}$ through the mode-generating system $(\mathfrak{G}_\beta^{\ddagger})$. It is obtained by evaluating the field at plane $(y^{\ddagger})$ created by a point source at plane (y), or vice versa.

The coupling factor could be obtained by substituting Eqs. (37) and (38) in Eq. (34) and integrating. This would be most easily done by integrating first the product of the respective mode-generating functions $E_\alpha(\mathbf{x};\mathbf{y})$ and $E_\beta^{\ddagger}(\mathbf{x};\mathbf{y}^{\ddagger})$ and expanding the result in power series of the four variables jky_1, jky_2, $-jky_1^{\ddagger}$, and $-jky_2^{\ddagger}$. Actually, it is not even necessary to perform the integration, if we recall that the coupling factor is independent of the plane of integration. To within a constant factor, the coupling between the two mode-generating functions is the field at plane $(y^{\ddagger})$ which transforms at plane (x) into a field identical to the field created by the point source S_α (see Fig. 3). Indeed, at plane $(y^{\ddagger})$ the field of beam β reduces to an impulse function.

To obtain the field at $(y^{\ddagger})$ equivalent to S_α, we have to evaluate the ray matrix $\mathfrak{M}$ corresponding to the mode-generating system $(\mathfrak{G}_\alpha)$ followed by the mode-generating system $(\mathfrak{G}_\beta^{\ddagger})$ taken from the output to the input plane.

With the help of Eq. (10), we get

$$\begin{aligned}\mathfrak{M}&=\mathfrak{M}_\beta^{\ddagger-1}\mathfrak{M}_\alpha\\&=\tfrac{1}{4}\begin{bmatrix} \tilde{\mathbf{P}}_\beta^{\ddagger} & -\tilde{\mathbf{Q}}_\beta^{\ddagger} \\ jk\tilde{\mathbf{P}}_\beta & -jk\tilde{\mathbf{Q}}_\beta \end{bmatrix}\begin{bmatrix} jk\mathbf{Q}_\alpha^{\ddagger} & \mathbf{Q}_\alpha \\ jk\mathbf{P}_\alpha^{\ddagger} & \mathbf{P}_\alpha \end{bmatrix}\\&=\tfrac{1}{4}\begin{bmatrix} jk(\tilde{\mathbf{P}}_\beta^{\ddagger}\mathbf{Q}_\alpha^{\ddagger}-\tilde{\mathbf{Q}}_\beta^{\ddagger}\mathbf{P}_\alpha^{\ddagger}) & \tilde{\mathbf{P}}_\beta^{\ddagger}\mathbf{Q}_\alpha-\tilde{\mathbf{Q}}_\beta^{\ddagger}\mathbf{P}_\alpha \\ (jk)^2(\tilde{\mathbf{P}}_\beta\mathbf{Q}_\alpha^{\ddagger}-\tilde{\mathbf{Q}}_\beta\mathbf{P}_\alpha^{\ddagger}) & jk(\tilde{\mathbf{P}}_\beta\mathbf{Q}_\alpha-\tilde{\mathbf{Q}}_\beta\mathbf{P}_\alpha) \end{bmatrix} \quad (39)\\&=\begin{bmatrix} (\beta^{\ddagger};\alpha^{\ddagger}) & (\beta^{\ddagger};\alpha)/jk \\ jk(\beta;\alpha^{\ddagger}) & (\beta;\alpha) \end{bmatrix}.\end{aligned}$$

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In the last expression in Eq. (39), we have introduced the generalized ray invariants defined in Eq. (15).

Substituting the expression obtained for $\mathfrak{M}$ in Eqs. (8) and (5), we get the field at plane $(y^{\ddagger})$

$$E(\mathrm{y}^{\ddagger};\mathrm{y})=|(\alpha;\beta^{\ddagger})|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}jk\tilde{\mathrm{y}}(\beta^{\ddagger};\alpha)^{-1}(\beta^{\ddagger};\alpha^{\ddagger})jk\mathrm{y} + jk\tilde{\mathrm{y}}(\beta^{\ddagger};\alpha)^{-1}jk\mathrm{y}^{\ddagger}-\tfrac{1}{2}jk\tilde{\mathrm{y}}^{\ddagger}(\beta;\alpha)(\beta^{\ddagger};\alpha)^{-1}jk\mathrm{y}^{\ddagger}], \quad (40)$$

where a constant factor has been dropped.

The exponential term in Eq. (40) is the generating function for Hermite polynomials in four variables whose explicit expression is given in the Appendix. The coefficients of the expansion of the right-hand side of Eq. (40) in power series of jky_1, jky_2, $-jky_1^{\ddagger}$, $-jky_2^{\ddagger}$ are, after normalization,

$$C_{\alpha\beta m_1m_2m_1'm_2'}=|(\alpha;\beta^{\ddagger})|^{-\frac{1}{2}}(m_1!m_2!m_1'!m_2'!)^{-\frac{1}{2}}\cdot \times He_{m_1m_2m_1'm_2'}(0;\mathfrak{N}), \quad (41)$$

where

$$\mathfrak{N}\equiv\begin{bmatrix}(\beta^{\ddagger};\alpha)^{-1}(\beta^{\ddagger};\alpha^{\ddagger}) & (\beta^{\ddagger};\alpha)^{-1}\\ -(\alpha;\beta^{\ddagger})^{-1} & (\beta;\alpha)(\beta^{\ddagger};\alpha)^{-1}\end{bmatrix}. \quad (42)$$

In the special case where $\alpha\equiv\beta$, we have, from Eq. (21),

$$\mathfrak{N}=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

The right-hand side of Eq. (41) reduces to $\delta_{m_1m_1'}\cdot\delta_{m_2m_2'}$, where $\delta_{m_im_i'}=1$ if $m_i=m_i'$, and zero otherwise. This demonstrates the biorthogonality of the sets of functions $E_{m_1m_2}(\mathrm{x})$ and $E_{m_1m_2}{}^{\ddagger}(\mathrm{x})$. If we are interested only in the propagation of beams through lossless systems, we can replace $\ddagger$ by a star, indicating complex conjugate values, and obtain the usual mode-orthogonality condition.

As an example of application of Eq. (41), let us evaluate the coupling factor between a beam α in the fundamental mode and a coaxial beam β in the mode m (m even), in two dimensions. Using Eqs. (41) and (A8), we get

$$C_{\alpha\beta 0m}=2^{-m/2}(m!)^{\frac{1}{2}}[(m/2)!]^{-1} \times(\beta^*;\alpha^*)^{m/2}(\alpha;\beta^*)^{-(m+1)/2}. \quad (43a)$$

Using the identity

$$(\beta^*;\alpha^*)(\beta;\alpha)=(\alpha;\beta^*)(\alpha^*;\beta)+(\alpha;\alpha^*)(\beta;\beta^*), \quad (43b)$$

and setting

$$\kappa^{-1}\equiv(\alpha^*;\beta)(\beta^*;\alpha)=(w_\alpha/w_\beta+w_\beta/w_\alpha)^2/4 +k^2w_\alpha{}^2w_\beta{}^2(1/R_\alpha-1/R_\beta)^2/16, \quad (43c)$$

where $w_{\alpha,\beta}$ and $R_{\alpha,\beta}$ denote the beam radii and wave-front radii of curvature, respectively, we find that the power coupling is

$$C_{\alpha\beta 0m}C_{\alpha\beta 0m}{}^*=2^{-m}m![(m/2)!]^{-2}\kappa^{\frac{1}{2}}(1-\kappa)^{m/2}, \quad (43d)$$

in exact agreement with an expression obtained by Kogelnik [Eq. (23d) of Ref. 1] by use of a direct integration technique involving finite hypergeometric series.

COUPLING BETWEEN MODES WITH DIFFERENT AXES

The result given in the last section can be generalized to the case of modes that have distinct axes. This is done by considering misaligned mode-generating systems.

When an optical system is misaligned, constant vectors are to be introduced in the ray equations, as shown in Eq. (16a).

The mode-generating matrices $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta{}^{\ddagger}$ are consequently of the form

$$\mathfrak{M}_\alpha=\begin{bmatrix} j\frac{k}{2}\mathrm{Q}_\alpha{}^{\ddagger} & \frac{1}{2}\mathrm{Q}_\alpha & \bar{\mathrm{q}}_\alpha\\ j\frac{k}{2}\mathrm{P}_\alpha{}^{\ddagger} & \frac{1}{2}\mathrm{P}_\alpha & \bar{\mathrm{p}}_\alpha\\ 0 & 0 & 1\end{bmatrix}, \quad (44)$$

$$\mathfrak{M}_\beta{}^{\ddagger}=\begin{bmatrix} -j\frac{k}{2}\mathrm{Q}_\beta & \frac{1}{2}\mathrm{Q}_\beta{}^{\ddagger} & \bar{\mathrm{q}}_\beta\\ -j\frac{k}{2}\mathrm{P}_\beta & \frac{1}{2}\mathrm{P}_\beta{}^{\ddagger} & \bar{\mathrm{p}}_\beta\\ 0 & 0 & 1\end{bmatrix}, \quad (45)$$

instead of Eqs. (35) and (36). Substituting these expressions in Eqs. (8), (17), and (2), we find that $(\bar{\mathrm{q}}_\alpha,\bar{\mathrm{p}}_\alpha)$ physically represents the axis of beam α and $(\bar{\mathrm{q}}_\beta,\bar{\mathrm{p}}_\beta)$ the axis of beam β.

Using the same procedure as before, we evaluate the coupling factor between the modes generated by $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta{}^{\ddagger}$ by first calculating the field at plane $(y^{\ddagger})$ equivalent to a point source at y. The point characteristic of the sequence of mode-generating systems is most conveniently obtained by evaluating the ray matrix $\mathfrak{M}_\beta{}^{\ddagger-1}\mathfrak{M}_\alpha$ and using Eqs. (8), (16b), and (17). This procedure is not applicable, however, to the total optical length of the optical axis, which must be evaluated directly. After expansion of the field at $\mathrm{y}^{\ddagger}$ in power series of jky_1, jky_2, $-jky_1^{\ddagger}$, $-jky_2^{\ddagger}$, the following expression for the coupling factor is obtained:

$$C_{\alpha\beta m_1m_2m_1'm_2'} =|(\alpha;\beta^{\ddagger})|^{-\frac{1}{2}}\exp[2(\beta^{\ddagger};\bar{\alpha}-\bar{\beta})\sim(\alpha;\beta^{\ddagger})^{-1}(\alpha;\bar{\alpha}-\bar{\beta})] \times\exp[-\tfrac{1}{2}(\bar{\alpha}-\bar{\gamma};\bar{\beta}-\bar{\gamma})](m_1!m_2!m_1'!m_2'!)^{-\frac{1}{2}} \times\mathrm{He}_{m_1m_2m_1'm_2'}(2\mathfrak{N}^{-1}\mathfrak{A};\mathfrak{N}), \quad (46)$$

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where $\mathfrak{A}$ denotes the four-dimensional vector

$$\mathfrak{A}\equiv\begin{bmatrix}(\beta^{\ddagger};\alpha)^{-1}(\beta^{\ddagger};\bar{\beta}-\bar{\alpha})\\ (\alpha;\beta^{\ddagger})^{-1}(\alpha;\bar{\beta}-\bar{\alpha})\end{bmatrix}. \quad (47)$$

The 4×4 matrix $\mathfrak{N}$ in Eq. (46) is defined in Eq. (42). $\bar{\gamma}$ represents a ray with components $\bar{\mathrm{q}}_\gamma$, $\bar{\mathrm{p}}_\gamma$, such that $\bar{\mathrm{q}}_\gamma=\bar{\mathrm{p}}_\gamma=0$ at plane (x). Equation (46) generalizes, for three dimensions and arbitrary mode numbers, an expression given before.[8]

Let us point out, in conclusion, that, besides its conceptual simplicity, the point-source representation of modes has the advantage that no integration is required to obtain the field transformation of modes and the coupling between modes.

APPENDIX: HERMITE POLYNOMIALS IN SEVERAL VARIABLES

An explicit expression for Hermite polynomials in N variables has been obtained by Grad[9] for the case where the quadratic form defining them reduces to a sum of squares. From this result, the general expression is easily obtained,

$$\mathfrak{H}\mathrm{e}^{(n)}(\mathrm{x};\mathrm{v})=\xi^n-\mathrm{v}\xi^{n-2}+\mathrm{v}^2\xi^{n-4}-\cdots +\begin{cases}\mathrm{v}^{n/2}, & n\text{ even}\\ (-\mathrm{v})^{(n-1)/2}\xi, & n\text{ odd.}\end{cases} \quad (A1)$$

In this expression, x denotes a vector with cartesian components x_i, $i=1,2\ldots N$, and v a symmetric tensor of order 2, with components ν_{ij}, i, $j=1,2\ldots N$. The vector ξ has components $\xi_i=\sum_j\nu_{ij}x_j$.

$\mathfrak{H}\mathrm{e}^{(n)}(\mathrm{x};\mathrm{v})$ is consequently defined as a tensor of order n. The components of $\mathfrak{H}\mathrm{e}^{(n)}$ that have the same number m_k of indices equal to k $(k=1,2\ldots N)$ are equal. We denote these components $\mathrm{He}_{m_1m_2\cdots m_N}(\mathrm{x};\mathrm{v})$. This is the notation used in the main text.

Note that a term such as $\xi^r\mathrm{v}^s$ in Eq. (A1) represents a tensor of order $r+2s$. Given $r+2s$ subscripts, the corresponding component of $\xi^r\mathrm{v}^s$ is a sum of terms, each a product of r ξ's and s v's; this sum is extended over all distinct terms.

In the case of one variable $(N=1)$, x, ξ, and v are scalars, and we have $n=m$ subscripts, all equal to 1. The terms in $\mathrm{v}\xi^{sm-2s}$ are therefore equal. We have only to count them. We get

$$\mathfrak{H}\mathrm{e}^{(n)}(\mathrm{x};\mathrm{v})\equiv He_m(x;\nu)=m!\sum_{s=0}^{[m/2]}\frac{(-\nu/2)^s\xi^{m-2s}}{(m-2s)!s!} \quad (A2)$$

$$=\nu^{m/2}He_m(\nu^{\frac{1}{2}}x),$$

where we have defined

$$He_m(x)=m!\sum_{s=0}^{[m/2]}\frac{(-\frac{1}{2})^sx^{m-2s}}{(m-2s)!s!}. \quad (A3)$$

Note, incidentally, that some authors use the polynomials

$$H_m(x)\equiv2^{m/2}He_m(2^{\frac{1}{2}}x). \quad (A4)$$

In two variables $(N=2)$, we have m_1 indices equal to 1 and m_2 indices equal to 2. The general term has therefore the form $\nu_{11}{}^\alpha\nu_{22}{}^\beta\nu_{12}{}^{\gamma-\alpha-\beta}\xi_1{}^{m_1-\gamma-\alpha+\beta}\xi_2{}^{m_2-\gamma-\beta+\alpha}$, the summation being carried over the indices α, β, γ. Starting, for instance, with ν_{12}, we see that there are m_1 ways of choosing the first index of ν among the m_1 indices 1 available, and m_2 ways of choosing the second index among the m_2 indices 2. The coefficient of ν_{12} is therefore m_1m_2. Once this choice has been made, there are m_1-1 indices left in group 1, and m_2-1 indices left in group 2. If we keep proceeding in that way, and divide the result by the number of identical products obtained, we find that $He_{m_1m_2}(\mathrm{x};\mathrm{v})$ assumes the form

$$He_{m_1m_2}(\mathrm{x};\mathrm{v})=\xi_1{}^{m_1}\xi_2{}^{m_2}-\left(\frac{1}{2}\frac{m_1(m_1-1)}{1}\nu_{11}\xi_1{}^{m_1-2}\xi_2{}^{m_2}+\frac{m_1m_2}{1}\nu_{12}\xi_1{}^{m_1-1}\xi_2{}^{m_2-1}+\frac{1}{2}\frac{m_2(m_2-1)}{1}\nu_{22}\xi_1{}^{m_1}\xi_2{}^{m_2-2}\right)+\cdots$$
$$+\sum_{\alpha,\beta=0}^{[\exp]}\frac{(-)^\gamma m_1!m_2!\nu_{11}{}^\alpha\nu_{22}{}^\beta\nu_{12}{}^{\gamma-\alpha-\beta}\xi_1{}^{m_1-\gamma-\alpha+\beta}\xi_2{}^{m_2-\gamma-\beta+\alpha}}{2^{(\alpha+\beta)}\alpha!\beta!(\gamma-\alpha-\beta)!(m_1-\gamma-\alpha+\beta)!(m_2-\gamma+\alpha-\beta)!}+\cdots. \quad (A5)$$

In Eq. (A5), [exp] indicates that the sum terminates when one of the exponents is equal to zero.

For $n=m_1+m_2$ even, the last term in the series is obtained when $\gamma=n/2$. This term is

$$He_{m_1m_2}(0;\mathrm{v})=\sum_{\alpha=0}^{\theta}\frac{(-)^{n/2}m_1!m_2!\nu_{11}{}^\alpha\nu_{22}{}^{\alpha-\theta}\nu_{12}{}^{m_1-2\alpha}}{2^{2\alpha-\theta}\alpha!(\alpha-\theta)!(m_1-2\alpha)!}, \quad (A6)$$

where we have set for brevity, $\theta=(m_1-m_2)/2$. It is assumed that $m_1\geq m_2$.

In the special case where $m_2=0$, $m_1\equiv m$, Eq. (A5) reduces to

$$He_{m0}(\mathrm{x};\mathrm{v})=\nu_{11}{}^{m/2}He_m(\nu_{11}{}^{-\frac{1}{2}}\xi_1), \quad (A7)$$

and, for $\mathrm{x}=0$ and m even,

$$He_{m0}(0;\mathrm{v})=\frac{m!(-\nu_{11}/2)^{m/2}}{(m/2)!}. \quad (A8)$$

Expressions for $He_{m_1m_2m_3m_4}(\mathrm{x};\mathrm{v})$ can similarly be obtained from Eq. (A1).

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