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Intensity noise of Kerr oscillators

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Absorbing or emitting elements generate noise waves. The main purpose of this paper is to determine from first principles the spectral density of noise waves relating to nonlinear elements. This was done by considering the combination of linear elements (whose noise properties are well understood) and lossless circuits that are nonlinear because of the Kerr effect. Lossless nonlinear circuits transform noise waves but do not generate noise. A semiclassical theory shows that noise waves remain at the shot noise level (for full population inversion) if the optical gain is considered a function of photon rate (rather than optical intensity). This result, in exact agreement with an independent theory of spectral-hole burning, is conjectured to be general. Intensity fluctuations of a Kerr oscillator are squeezed below the shot-noise level for large Kerr constants.

1. Introduction

An oscillator of any kind consists essentially of an emitter and an absorber. Oscillation is stable when the rate at which photons are emitted is a sublinear function of optical intensity if the absorber is linear, or when the rate at which photons are absorbed is a superlinear function of optical intensity if the emitter is linear. Under those conditions, noise can be treated as a small perturbation from the steady state.

In most laser oscillators, the sublinear behaviour of emitters results from a reduction of the number of atoms in the upper state as intensity grows. We investigate a different kind of laser oscillator in which both the emitter and the absorber are linear. They are interconnected through a lossless nonlinear circuit that ensures stability. The noise properties of linear elements are well understood. The circuit nonlinearity results from the Kerr effect, which introduces phase shifts proportional to light intensity variations [1].

The rate R at which photons are absorbed will be shown to be of the form

$$R = G(R)P + \mathrm{r}' \qquad \mathrm{S}_{rr} = \eta R \qquad \eta \equiv (N_1 + N_2)/(N_1 - N_2) \tag{1}$$

where P denotes the (modulus) square of the voltage across the element at optical frequency ν. The nonlinearity occurs through the dependence of the conductance G on rate R. The expression of the (double-sided) spectral density S_{rr} of the noise term r' in Equation 1, where N_1, N_2 denote the atomic populations in the lower and upper state, respectively, implies that r' is at the shot-noise level when either $N_1 = 0$ (ideal emitter) or $N_2 = 0$ (cold absorber), irrespective of the nonlinearity, i.e. of the dependence of G on R. Because η and

R are both positive for an absorber and both negative for an emitter, the product ηR is positive. Nonessential noise sources due, e.g., to spontaneous recombination in modes other than the oscillating mode, are not considered.

It is remarkable that Equation 1 also applies to laser diodes affected by spectral-hole burning [2]. Photonic flows are regulated by external resistances and slow diffusion of carriers within each band, the two acting in series. We thus conjecture that Equation 1 gives the least noise associated with nonlinear elements. Let us emphasize that the additional noise term r' would not be at the shot-noise level if G were considered to depend on P rather than on R [3] (under the $N_1 = 0$ or $N_2 = 0$ conditions).

For illustration, consider a saturable absorber [4], for which

$$G(R) = G_o - R/P_s \tag{2}$$

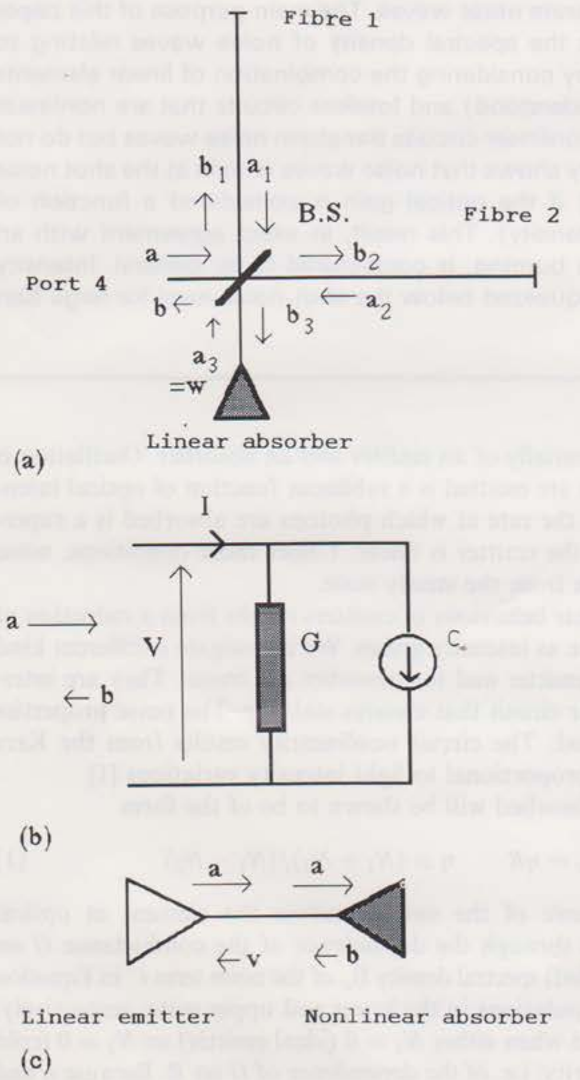

Figure 1 (a) The nonlinear absorber consists of a 50% beam splitter (B.S.) and two fibres terminated by mirrors and exhibiting Kerr constants of opposite signs, in ports 1 and 2. A linear matched absorber is in port 3. Port 4 is the useful port (label 4 omitted for simplicity). *a*-Waves are entering the B.S., while *b*-waves are exiting the B.S. (b) Schematic of an element with Nyquist-like current source c. (c) Kerr's oscillator, with linear emitter (white triangle) and superlinear absorber (grey triangle). The latter is the configuration in (a), as viewed from port 4. The tuned circuit is not shown.

where G_o and P_s are positive constants. Equation 1 can be written alternatively in that case as

$$R = [G_o/(1 + P/P_s)]P + r'/(1 + P/P_s) \tag{3}$$

where the conductance, or loss, is now expressed as a function of P rather than R. Notice that the spectral density of the noise term in Equation 3 is not equal to that of r' in Equation 1, but depends on the nonlinearity.

The specific arrangement shown in Fig. 1a consists of a 50% ideal beam splitter (BS), two optical fibres having opposite Kerr constants in ports 1 and 2, and a linear matched absorber terminating port 3. Port 4 (label 4 suppressed for simplicity) is the useful port. We will demonstrate the following:

(1) In the steady-state, port 4 behaves as a matched load, i.e. as a pure conductance G, equal to the characteristic conductance G_c of the transmission line.
(2) G is independent of the optical frequency to first order, i.e. the loss is nondispersive.
(3) When the input light intensity varies, G varies but remains a pure conductance.
(4) The output noise wave ($b_4 \equiv b$) coincides with the noise wave $a_3 \equiv \mathrm{W}$ generated by the absorber.

The arrangement in Fig. 1a was selected in order that conditions (1) to (3) are fulfilled, while (4) is the basic finding of this paper.

The relationship between circuit and wave formalisms is clarified in Section 2. The arrangement is treated without noise in Section 3 and a nonlinearity factor κ is introduced. It is proved in Section 4 that the noise wave generated by the absorber is unaffected by the lossless nonlinear device. Application to oscillators is made in Section 5. The conclusion is presented in Section 6.

Upper bars indicating steady-state values are omitted when no confusion with instantaneous values may arise. Deviations from steady-state values are denoted by δ. For any complex number z, we set $z \equiv z' + \mathrm{i}z''$, and $|z|^2 \equiv z'^2 + z''^2$. Re( ) and Im( ) denote real and imaginary parts, respectively.

2. Circuit and wave formalism

The purpose of this section is to determine wave amplitude variations when the transmission line is terminated by a (possibly nonlinear) conductance G.

Let $V(2h\nu)^{1/2}$ and $I(2h\nu)^{1/2}$ denote, respectively, the voltages and currents at frequency ν, the sign convention being shown in Fig. 1b. The absorbed photon rate R, defined as the ratio of dissipated electromagnetic power and photon energy ($h\nu$) is the real part of V^*I. Optical intensity is defined as $P \equiv V^*V$. The steady-state value of V is assumed real.

A complex Nyquist-like noise current c is associated with the conductance G as shown in Fig. 1a. From Ohm's and Kirchhoff's laws

$$I = GV + \mathrm{c} \tag{4}$$

Multiplying both sides of Equation 4 by V^* and taking the real part,

$$R = \mathrm{Re}(V^*I) = GP + r', \qquad P \equiv |V|^2 \qquad r' \equiv \mathrm{Re}(V^*\mathrm{c}) = V\mathrm{c}' \tag{5}$$

First-order variations of R and P are thus related by

$$(1-\kappa)\delta R/R = \delta P/P + r'/R \qquad \kappa \equiv (R/G)(\mathrm{d}G/\mathrm{d}R) \tag{6}$$

if we consider that G depends on R only. We have introduced in Equation 6 the dimensionless nonlinearity factor κ. Since we are looking for a superlinear dependence of the absorbed rate R on P, κ should be positive.

Consider now the wave formalism. Incident and reflected wave amplitudes are denoted by a and b, as usual. Setting for simplicity the characteristic admittance G_c as unity,

$$V = a + b \qquad I = a - b \qquad R = |a|^2 - |b|^2 \qquad P = |a + b|^2 \tag{7}$$

If $G = 1$ in the steady state, the steady-state value of the reflected wave b vanishes, and thus $\delta b \equiv b$. Since V, and thus I and a, are real in the steady state, the first-order variations of R and P are

$$\delta R = a 2\delta a' \qquad \delta P = a(2\delta a' + 2\delta b') \tag{8}$$

Substituting the expressions in Equation 8 into Equation 6, the relation between $\delta a'$ and $\delta b'$ reads

$$\kappa \delta a' + \delta b' = \mathsf{w}' \qquad \mathsf{w}' \equiv -\mathsf{r}'/2a \tag{9}$$

For a linear absorber ($\kappa = 0$), Equations 6 and 9 read, respectively,

$$\delta R/R = \delta P/P + \mathsf{r}'/R \qquad \delta b' = \mathsf{w}' \tag{10}$$

A simple argument provides the spectral density of r'. Assume that all the atoms are in the ground state, and consider the situation where $\delta P = 0$, i.e. the atoms are submitted to an optical field that cannot vary. (This situation is referred to in quantum mechanics as the photon number-state). It is then intuitive that the atomic transitions from the ground state to the excited state are independent, since the atoms cannot 'communicate' with one another through induced field fluctuations, and the atomic wavefunctions do not overlap. Accordingly, the fluctuations of R must be at the shot-noise level in that situation. Since $\delta P = 0$ this is also the case for r', and thus

$$\mathsf{S}_{\mathsf{r}'} = R \Rightarrow \mathsf{S}_{\mathsf{w}'} = 1/4 \tag{11}$$

This also follows from the well-known expression for Nyquist-like current spectral densities including Planck's zero-point fluctuation, noting that the relative fluctuations of V in Equation 5 are negligible [3]. Because w' is the real part of a narrow-band process w, w' and the imaginary part w'' are independent and both have spectral densities of 1/4 at $T = 0$ K, but this fact is not needed in the following. If there are on the average N_1 atoms in the ground state and N_2 atoms in the excited state, the more general result in Equation 1 applies. (The noise waves w', w'' emitted by the cold linear absorber are sometimes called 'vacuum fluctuations'. For squeezed vacua, the spectral densities of w' and w'' would no longer be given by Equation 11, but this situation is not considered here.)

There is no obvious reason why the emitted noise wave should remain at the shot-noise level when the element is nonlinear. The purpose of the following sections is to establish that this is nevertheless the case, provided the conductance is considered a function of the emitted rate R rather than of the optical intensity P.

3. Nonlinearity factor

Properties (1) to (3) in Section 1 are shown to apply to the configuration in Fig. 1a. The noise terms are not considered here.

The basic property of a 50% ideal beam splitter (BS) is that wave amplitudes are multiplied by $1/\sqrt{2}$ upon transmission and $\mathrm{i}/\sqrt{2}$ upon reflection, unimportant phase factors being omitted. It is easy to verify that the scattering matrix of this 4-port device is symmetrical (reciprocal circuit) and unitary (conservative circuit). The a- and b-waves are entering and leaving the BS, respectively. For the absorber in port 3, the b- and a-waves introduced in Section 2 are relabelled, respectively, a_3 and b_3.

Thus, referring to Fig. 1a, the wave b reflected from port 4 is

$$\sqrt{2}b = \mathrm{i}a_1 + a_2 \tag{12}$$

Let the round-trip phase shifts introduced by fibres 1 and 2 be denoted $\phi_1(\nu) + \delta\phi_1 - \pi/2$ and $\phi_2(\nu) + \delta\phi_2 - \pi/2$, respectively. The phase shifts ϕ_1 and ϕ_2 are taken to be multiples of 2π at the operating frequency. Accordingly, for a small frequency variation $\delta\nu$,

$$a_1 = -\mathrm{i}b_1 \exp\left[\mathrm{i}(\mathrm{d}\phi_1/\mathrm{d}\nu)\delta\nu + \mathrm{i}\delta\phi_1\right] \tag{13a}$$

$$a_2 = -\mathrm{i}b_2 \exp\left[\mathrm{i}(\mathrm{d}\phi_2/\mathrm{d}\nu)\delta\nu + \mathrm{i}\delta\phi_2\right] \tag{13b}$$

Since the absorber on port 3 is linear and matched, $a_3 = 0$ and

$$\sqrt{2}b_1 = \mathrm{i}a \qquad \sqrt{2}b_2 = a \tag{14}$$

where a denotes the input wave in port 4.

Collecting Equations 12 to 14, to first order in $\delta\phi$,

$$\begin{aligned} 2b &= \mathrm{i}a\left\{\exp\left[\mathrm{i}(\mathrm{d}\phi_1/\mathrm{d}\nu)\delta\nu + \mathrm{i}\delta\phi_1\right] - \exp\left[\mathrm{i}(\mathrm{d}\phi_2/\mathrm{d}\nu)\delta\nu + \mathrm{i}\delta\phi_2\right]\right\} \\ &\approx a(\delta\phi_2 - \delta\phi_1) \end{aligned} \tag{15}$$

assuming that the two fibres have the same delays: $\mathrm{d}\phi_1/\mathrm{d}\nu = \mathrm{d}\phi_2/\mathrm{d}\nu$.

Equation 15 shows that the reflected wave b vanishes in the steady state, i.e. the system appears as a matched load as asserted in (1) in Section 1. It is unaffected by a small change of the optical frequency ν as asserted in (2). Finally b/a is real, expressing the fact that the conductance remains real, as asserted in (3).

Consider now the phase shift ϕ experienced by a wave propagating in a fibre exhibiting the Kerr effect. The phase shift deviates by an amount $\delta\phi$ proportional to the input-wave intensity change:

$$\delta|a|^2 = 2|a|^2 \mathrm{Re}(\delta a/a) \tag{16}$$

For fibre 1 with input wave b_1,

$$\delta\phi_1 = \kappa\,\mathrm{Re}(\delta b_1/b_1) = \kappa\,\delta a'/a \tag{17}$$

where κ is proportional to the Kerr constant. Indeed, according to Equation 14, $\delta b_1/b_1 = \delta a/a$, and the steady-state value of the input wave a is real.

Fibre 2 is assumed to exhibit the same Kerr constant as fibre 1, but with opposite sign. Accordingly, using Equation 14 again,

$$\delta\phi_2 = -\kappa\,\mathrm{Re}(\delta b_2/b_2) = -\kappa\,\delta a'/a \tag{18}$$

Introducing Equations 17 and 18 to Equation 15,

$$b = -\kappa\,\delta a' \Rightarrow \kappa\,\delta a' + \delta b' = 0 \qquad \delta b'' = 0 \tag{19}$$

since κ is real.

Comparison of Equations 19 and 9 without the noise term reveals that the configuration in Fig. 1a is equivalent to a nonlinear conductance with nonlinearity parameter κ as defined in Equation 6.

4. Noise waves

The linear absorber in port 3 generates a noise wave w whose spectral density was given in Equation 11. This noise wave coincides with the wave a_3 entering the beam splitter. In spite of transformations due to the beam splitter and propagation through nonlinear optical fibres, this wave eventually exits unaffected from port 4, as Equation 25 will show. Equations 12 and 13 are unchanged, while Equation 14 becomes

$$\sqrt{2}b_1 = ia + w \qquad \sqrt{2}b_2 = a + iw \tag{20}$$

Equation 15 becomes

$$2b \approx a(\delta\phi_2 - \delta\phi_1) + 2w \tag{21}$$

Equations 17 and 18 now read

$$\delta\phi_1 = \kappa \operatorname{Re}(\delta b_1/b_1) = \kappa(\delta a' + w'')/a \tag{22}$$

$$\delta\phi_2 = -\kappa \operatorname{Re}(\delta b_2/b_2) = -\kappa(\delta a' - w'')/a \tag{23}$$

and thus

$$\delta\phi_1 - \delta\phi_2 = 2\kappa\,\delta a'/a \tag{24}$$

coincides with the previous result, Equations 17 and 18.

Finally, from Equations 24 and 21, the relations

$$\delta b = -\kappa\,\delta a' + w \Rightarrow \delta b' + \kappa\,\delta a' = w' \qquad S_{w'} = 1/4 \tag{25}$$

demonstrate (4) in Section 1.

Thus, the configuration in Fig. 1a behaves as a matched load in the steady state. The reflected b-wave is the sum of a term proportional to the deviation of the input-wave intensity from its steady-state value and a noise wave that coincides with the one generated by the linear absorber in port 3.

The result in Equation 25 can be shown to be general when conditions (1) to (3) in Section 1 are fulfilled.

5. Intensity noise of a Kerr oscillator

It was established in previous sections that the configuration in Fig. 1a behaves as a superlinear absorber (for $\kappa > 0$). To construct an oscillator, it suffices to connect to port 4 a linear negative conductance equal to -1 (remembering that the characteristic conductance is taken as unity, for simplicity). A constant negative conductance can be realized by applying a constant-voltage drive to a laser diode. The tuned circuit need not be specified as long as only low-frequency intensity noise is considered.

Equation 25 with $\kappa = 0$ applies to negative linear conductances, but a is now the wave leaving the emitter, while b is the wave entering that element. The noise wave entering the emitter, denoted by v$'$, is independent of w$'$ and has the same spectral density (1/4), assuming for simplicity complete population inversion.

At low baseband frequencies the wave exiting the emitter coincides with the wave entering the absorber, and conversely, see Fig. 1c. Thus

$$\delta b' + \kappa\,\delta a' = w' \qquad S_{w'} = 1/4 \quad \text{(nonlinear absorber)} \tag{26a}$$

$$\delta b' = v' \qquad S_{v'} = 1/4 \quad \text{(linear emitter)} \tag{26b}$$

The real part $\delta a'$ of the fluctuation of the wave a propagating from the emitter to the absorber is, from Equation 26,

$$\kappa\,\delta a' = w' - v' \tag{27}$$

The (double-sided) spectral density of the fluctuation $\delta|a|^2 = 2a\,\delta a'$ of the photon rate $R \equiv |a|^2$ flowing from the emitter to the absorber is therefore

$$R^{-1}S_{\delta R} = 2/\kappa^2 \tag{28}$$

This result shows that the oscillator output is amplitude-squeezed [5] (i.e. $S_{\delta R} < R$) if $\kappa > \sqrt{2}$, that is if the Kerr constant is sufficiently large.

6. Conclusion

With a view to establishing a semiclassical theory of noise for nonlinear absorbers or emitters, we have analysed a configuration consisting of a linear element, whose noise properties are well understood, connected to a lossless nonlinear circuit (Kerr medium). The configuration is equivalent to a pure conductance whose value depends on the input wave intensity but not on frequency. It is concluded that the noise wave remains at the shot-noise level, provided the optical gain is considered a function of photon rate (rather than of optical intensity). According to our formulation, the amplitude noise of Kerr's oscillators is below shot-noise for sufficiently large Kerr constants.

References

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