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Degenerate Optical Cavities

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An optical cavity is degenerate when an arbitrary ray retraces its own path after a single round trip. The condition for degeneracy is given for ring type cavities incorporating internal lenses, using geometrical optics methods. The simplest linear configurations require a spherical mirror or a corner cube, a thin lens, and a plane mirror. Planar rings with four plane mirrors require at least three thin focusing elements. A nonplanar ring is discussed which requires only two thin lenses. The alignment of degenerate cavities is, in general, as critical as the alignment of plane Fabry-Perot.

I. Introduction

From a geometrical optics point of view, an optical cavity is degenerate when an arbitrary ray retraces its own path after a single round trip. From this property, it is easy to show that any field configuration reproduces itself after a round trip within the approximation of the scalar Fresnel diffraction theory.

These optical cavities have applications in active imaging,[1] spatial scanning of lasers,[2] and regenerative amplification of distorted optical signals.* Since there is no need for mode matching with an external signal as in ordinary cavities, they are also useful as scanning interferometers.

A linear degenerate cavity comprised of two identical spherical mirrors and a confocal internal lens was originally proposed by Pole.[3] This cavity was subsequently generalized for end mirrors of unequal curvatures by Hardy.[1] An equivalent structure comprised of two identical confocal lenses and plane end mirrors has also been proposed.[4]

di Francia[5] pointed out that cavities, in which any ray retraces its path after more than one round trip, can also be considered degenerate if an off-axis ray is taken as the optical axis. From this point of view, the classical confocal cavity[6] is degenerate for far off-axis rays. The operation of such cavities, however, is strongly affected by aberrations.

With the help of the well-known theory of open resonators, based on gaussian modes[7] or geometrical optics,[8,9] we will point out the general features of ring type and linear type degenerate cavities. The results are subsequently applied to particular configurations in a search for degenerate cavities which are either simpler than those proposed before or more flexible in the fulfillment of the degeneracy condition. For clarity, two classes of degenerate cavities are distinguished: (1) plane rings with an even number of plane mirrors; and (2) nonplanar rings which have the peculiarity of introducing an image rotation equal to π about the optical axis. Examples of aberration-free degenerate cavities are also given.

II. General Properties of Degenerate Cavities

We restrict ourself to cavities that can be analyzed by considering paraxial rays lying in two mutually perpendicular meridional planes. The discussion which follows applies to either one of these two planes. We also assume that the reflecting mirrors are plane. No loss of generality results from this assumption, since a spherical mirror of radius R under normal incidence is equivalent to a plane mirror and a lens of focal length $f = R$ in front of it (notice that the lens is crossed twice by the optical axis in a round trip). More generally, a curved mirror is equivalent to a plane mirror followed by an astigmatic lens,* if the incidence plane coincides with a principal plane of the mirror surface.

A. Ring Type Degenerate Cavities

Let x_1, $\dot{x}_1$ define, respectively, the position and slope of a ray at an arbitrary plane perpendicular to the optical axis, and x_0, $\dot{x}_0$ the position and slope that this ray assumes at the same plane after a round trip.

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* The present study was motivated by a proposal by R. Kompfner to amplify narrow band distorted optical signals before detection to enhance optical receiver sensitivities. This scheme is discussed in Ref. 15.

* By astigmatic lens, we understand a lens which has different focal lengths in two perpendicular meridional planes, and by stigmatic lens a lens which has a cylindrical symmetry.

Within the first order approximation, these quantities are linearly related:

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}; \tag{1}$$

A, B, C, D in Eq. (1) are the elements of the round trip ray matrix. They satisfy the relation $AD - BC = 1$.

From Eq. (1), it is clear that any ray retraces its path ($x_1 = x_0$ and $\dot{x}_1 = \dot{x}_0$ for any value of x_0 and $\dot{x}_0$) when, and only when, the ray matrix is unity. This condition is expressed by

$$B = C = 0,$$
$$A = D = 1. \tag{2}$$

The condition $B = 0$ implies that the reference plane is imaged into itself; $A = 1$, that the magnification is unity; $C = 0$, that the transformation is telescopic. The scalar law of imaging for Fresnel diffraction[7] shows that any transverse field configuration is exactly reproduced after a round trip except for a constant phase shift equal to 0 (mod 2π) at the resonance frequency.

Upon examination of Eqs. (1) and (2), the following conclusions are reached: (1) a sufficient condition for degeneracy is that two rays which do not cross the optical axis at the same point retrace their own path after a round trip; and (2) three parameters (for instance the separations between four lenses on a ring of fixed total length) are required to fulfill the degeneracy condition.

B. Linear Degenerate Cavities

Let us consider now the special case of linear cavities limited by two plane end mirrors M_1, M_2, and call $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the ray matrix from M_1 to M_2. The round trip ray matrix, from M_1 to M_2 and back to M_1, is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & b \\ c & a \end{bmatrix} = \begin{bmatrix} ad + bc & 2ab \\ 2cd & ad + bc \end{bmatrix}. \tag{3}$$

From Eq. (2) and the relation $ad - bc = 1$, we see that a linear cavity is degenerate when, and only when, $b = c = 0$. The condition $b = 0$ indicates that one end mirror is imaged into the other. This conclusion applies also, of course, to the case of spherical end mirrors. The condition $c = 0$ indicates that the transformation from M_1 to M_2 is telescopic. The magnification, however, may not be unity. As a consequence, only two free parameters are necessary to fulfill the degeneracy condition.

In Sec. II. C we show that the degenerate cavities belong to a broader class of optical cavities which are *mode degenerate*.

C. Resonance Frequency Based on Mode Theory

From the gaussian mode theory, the resonance frequencies of a ring type cavity which has a round trip ray matrix spur equal to $A + D$ in any meridional plane and a round trip path length L are given by*

$$kL - (2p + l + 1)\cos^{-1}\frac{A + D}{2} = 2K\pi, \tag{4}$$

where k is the free propagation constant, p, l are, respectively, the radial and azimuthal mode numbers, and K is an integer. For a linear cavity, from Eq. (3), the argument of $\cos^{-1}$ in Eq. (4) becomes $ad + bc$. Equation (4) shows that the resonance frequencies depend neither on p nor on l when $A + D = 2$ (or $bc = 0$ for linear configurations). These cavities, however, are not generally degenerate because no mode of finite extend exists, unless $A = D = 1$, $B = C = 0$, in which case any field configuration can be viewed as a mode. A condition equivalent to $A + D = 2$ is that any ray going through a particular point on the optical axis, which may be called the center of the cavity, retraces its path after a round trip. This is clearly the case for any cavity having concentric spherical refracting and reflecting surfaces, and for plane parallel Fabry-Perot.

D. Alignment of Degenerate Cavities

All the cavities satisfying the condition $A + D = 2$ have the common property of requiring accurate alignments of their optical elements. The reason is that no optical axis (defined as a ray which retraces its own path) exists when the cavity is misaligned. This is readily seen by noting that the self-consistency ray equations,

$$x = Ax + B\dot{x} + \delta,$$
$$\dot{x} = Cx + D\dot{x} + \gamma, \tag{5}$$

(where δ and γ represent misalignments at the reference plane), have no solution when $(A - 1)(D - 1) - BC = 0$. This condition is equivalent to $A + D = 2$. How critical the alignment really is depends on the excitation field and the finesse of the cavity.

E. Half-degenerate Cavities

It is also of interest to discuss the case in which $A = D = -1$, $B = C = 0$ (or $a = d = 0$ for linear configurations). Equation (4) shows that the even modes (l even) and the odd modes (l odd) resonate at frequencies separated by half the free spectral range for any value of p.

We readily see that the self-consistency equation for the complex beam parameter q (Ref. 7),

$$q = (Aq + B)/(Cq + D), \tag{6}$$

is satisfied in that case for any value of q. In other words, any incident on-axis gaussian beam is matched to these cavities.* An arbitrary field is not faithfully transmitted, however, since the odd modes are dropped when the cavity is tuned at an even mode frequency.

Since the round trip ray matrix is equal to $-[1]$, the ray matrix for two round trips is unity. Accordingly, this type of cavity is degenerate if an arbitrary off-axis ray (which makes two round trips before closing on itself) is taken as the optical axis. This conclusion applies only if the coupling between the beams associated with the two turns of the optical axis is negligible. The alignment of such degenerate cavities is not critical but the field of view is strongly limited by aberrations. Notice also that half of the optical power is lost upon reflection on the semitransparent input mirror unless it is made fully reflective on half of its surface.

F. Resonance Frequency of Arbitrary Degenerate Cavities

The resonance condition of degenerate cavities incorporating astigmatic lenses and polarization dependent elements is readily obtained from geometrical optics. The total phase shift experienced by a ray in a round trip is equal to 0 (mod 2π) at a resonance frequency. Considering a paraxial ray pencil, the resonance frequencies are consequently given by

$$kL - \pi\mu + i\log(\psi_j) = 0 \pmod{2\pi}, \quad j = 1,2, \tag{7}$$

where the ray path length L is equal to the optical axis path length from the Fermat's principle; 2μ is the number of tangential and sagittal foci encountered by the ray pencil in a round trip†, and ψ_j, $j = 1,2$ are the polarization matrix eigenvalues. Values of μ and ψ_j will be given in the following sections for specific configurations. Notice that μ does not depend on the particular ray pencil considered. A simple argument can be given to show that the number of foci encountered between an object and an image plane by a ray pencil lying in a meridional plane is a fixed integer. Consider two rays limiting a ray pencil and take their positions at the object plane, and, consequently, at the image plane, as fixed. We see by continuity that these two rays necessarily cross a fixed number of times between the two planes, whatever their original slopes may be, because they can, at no point, be tangent to each other (if they were, they would coincide everywhere).

III. Plane Rings with Stigmatic Lenses and an Even Number of Plane Mirrors

Let us now discuss the degeneracy of a plane ring cavity with an even number of plane mirrors. If θ_1, θ_2, . . ., θ_{2N} are the angles between successive reflecting planes, a closed path generally exists when $\theta_1 + \theta_3 + \ldots + \theta_{2N-1} = 0 \pmod{\pi}$. The optical transformation is then a translation which leaves unchanged the position of off-set rays with respect to the optical axis. Accordingly, we may consider only the ray transformation resulting from the focusing elements, and find the condition that the ray matrix equal $+[1]$.

The ray matrix relating the position and slope of a ray at the image focal plane of a (possibly thick) lens of focal length f to the values taken at a distance c from the object focal plane is[10]

$$\begin{bmatrix} 0 & f \\ -(1/f) & 0 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -(1/f) & (c/f) \end{bmatrix}. \tag{8}$$

Let us consider two optical elements of focal lengths f_1 and f_2 on a closed path. It is easily seen, using twice the expression given in Eq. (8) that the condition for the round trip ray matrix to be equal to $+[1]$ is that the two optical elements are confocal and $f_1 + f_2 = 0$. These conditions cannot be satisfied with two thin lenses only, since the total path length would be $L = 2f_1 + 2f_2 = 0$.

A. Three Lens Degenerate Cavities

When there are three lenses of focal lengths f_1, f_2, f_3 along the closed path, the optical separations c_{12}, c_{23}, c_{31}, taken between adjacent foci as shown in Fig. 1(a), are required to be

$$c_{12} = \frac{f_1 f_2}{f_3}, \quad c_{23} = \frac{f_2 f_3}{f_1}, \quad c_{31} = \frac{f_3 f_1}{f_2}. \tag{9}$$

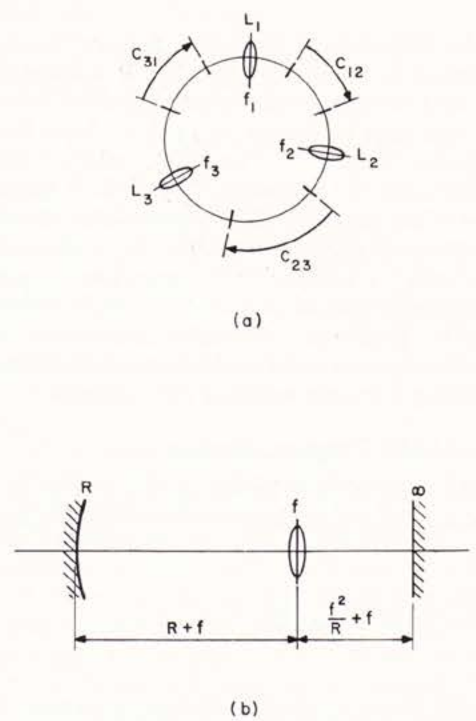

Fig. 1. Three-lens cavity: (a) shows the notations used for the calculation of the lens ray matrix; (b) is the degenerate linear configuration which may be obtained from (a) when $f_1 \equiv f_2 \equiv f$, and $c_{31} = c_{23} = f_3 \equiv R/2$.

* The on-axis field of a ray *pencil*, limited to a ray $x(z)$, is proportional to $x(z)^{-1}$. This expression, with $x(z)$ complex, is also applicable to fundamental gaussian beams ($p = l = 0$). The on-axis phase shift experienced by such a beam through an optical system described by Ed. (1) is, consequently, kL + phase of $(x_1/x_0) = kL$ + phase of $(A + B/q)$, where $q \equiv x_0/\dot{x}_0$ is the input complex beam parameter. For the case of a resonating cavity, Eq. (4) is obtained by substituting for q the solution of Eq. (6) (matched beam). The case of linear cavities was discussed in Ref. 7.

* Consequently, the end mirror surfaces of the well-known confocal cavity[6] do not coincide, in general, with phase fronts of the resonating mode, and the resonating field is not, in general, a pure standing wave.

† The term $-\pi\mu$ which expresses the phase anomaly of a ray pencil at a focus cannot be overlooked because μ may well be an odd number in the case of astigmatic lenses. Notice that Eq. (7) is also applicable to nonorthogonal optical systems.

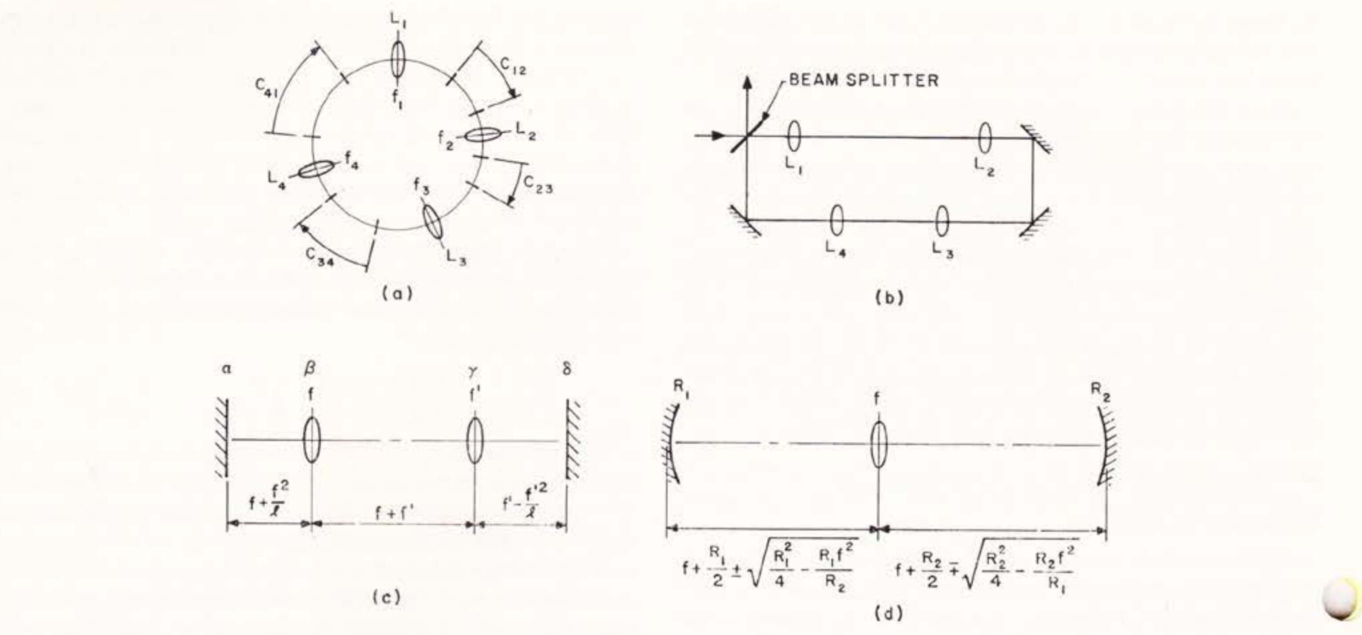

Fig. 2. Four-lens cavity: (a) shows the notations used for the calculation of the lens ray matrix; (b) shows a ring configuration. By adjusting the axial position of the lenses, an exact fulfillment of the degeneracy condition can be obtained, in general; (c) and (d) show the two degenerate linear configurations that a general four-lens cavity may take.

The total path length, under the above conditions of Eq. (9), is

$$L = 2(f_1 + f_2 + f_3) + c_{12} + c_{23} + c_{31} = \frac{(f_1f_2 + f_2f_3 + f_3f_1)^2}{f_1f_2f_3}. \tag{10}$$

From the requirement that the distance between adjacent lenses is a positive quantity, it follows that the three lenses must be convergent. If $f_2 = f_1 \equiv f$, the optical axis may be folded on itself to form the linear configuration shown in Fig. 1(b), which includes a spherical mirror of radius $R \equiv 2f_3$, a lens of focal length f, and a plane mirror. The separations between the optical elements given in Fig. 1(b) are readily obtained from Eq. (9). This cavity is somewhat simpler than the cavities proposed before,[1–4] which require two spherical end mirrors. We also notice that the degeneracy condition can be fulfilled for any positive values of R and f by a proper choice of the distances.

B. Four Lens Degenerate Cavities

For the case of four lenses shown in Fig. 2(a), the required optical separations between adjacent lenses are given by

$$\frac{f_4}{f_2} c_{12}c_{23} = -\frac{f_1}{f_3} c_{23}c_{34} = \frac{f_2}{f_4} c_{34}c_{41} = -\frac{f_3}{f_1} c_{41}c_{12} = f_2f_4 - f_1f_3. \tag{11}$$

Since only three of the above four equations are independent, we may choose arbitrarily the focal lengths and the total path length $L = 2(f_1 + f_2 + f_3 + f_4) + c_{12} + c_{23} + c_{34} + c_{41}$, and calculate the values which should be given to the optical separations from Eq. (11) and the above expression for L.* A ring cavity of this type, using four plane mirrors, is shown in Fig. 3(b).

Let us consider the special case in which $f_1 = f_2 \equiv f$ and $f_3 = f_4 \equiv f'$. A solution of Eq. (11) is $c_{23} = c_{41} = 0$ and $c_{12} + c_{34}(f/f')^2 = 0$. For that solution, the cavity can be folded on itself; it is equivalent to the linear cavity, shown in Fig. 2(c), which incorporates two lenses and two plane end mirrors. From the expression of the distances given in this figure (where l is an arbitrary length), it is clear that the optical transformation due to the internal lenses is telescopic. It may also be verified that the end mirrors are situated at conjugate points.

In order to reduce the cavity losses, it is of interest to replace the internal lenses by internal reflecting spherical mirrors. An obvious difficulty is that the internal mirrors (working under rather large incidence angles) present an important astigmatism: the focal lengths of a spherical mirror of radius R are[9] $(R/2)\cos\varphi$ and $(R/2)/\cos\varphi$ in the incidence plane and in the perpendicular plane, respectively, if φ is the incidence angle. A simple solution, however, can be found.

Let us suppose that two spherical mirrors of equal radii R are introduced at points β and γ of the configuration shown in Fig. 2(c) in place of the two lenses. If the plane defined by the points α, β, γ (which are no longer aligned) is perpendicular to the plane defined by β, γ, δ, a ray incident on the mirror β in the plane of incidence is perpendicular to the plane of incidence on the mirror γ, and reciprocally. Let us take the incidence angles at β and γ as both equal to φ. If the distances $\alpha\beta$ and $\gamma\delta$ are made equal by a proper choice of the parameter l, the exchange of $f = (R/2)\cos\varphi$ and $f' = (R/2)/\cos\varphi$ is immaterial and the degeneracy condition can be satisfied simultaneously in the two mutually perpendicular meridional planes. The condition defining l which has just been introduced is, from the expressions given in Fig. 2(c),

$$\frac{R}{2}\cos\varphi + \frac{[(R/2)\cos\varphi]^2}{l} = \frac{R}{2\cos\varphi} - \frac{(R/2\cos\varphi)^2}{l}. \tag{12}$$

If we take, for instance, $\varphi = \pi/4$, we get $l \simeq 1.76R$. The distance between the two spherical mirrors must be $\simeq 1.06R$ and the distance between a spherical mirror and the closest end mirror $\simeq 0.42R$.

The four-lens cavity may also take the form shown in Fig. 2(d) with two spherical end mirrors of radii R_1, R_2, and a single lens of focal length f. This cavity is more general than the one proposed by Hardy,[1] who considered only the case in which $1/R_1 + 1/R_2 = 1/f$. When this last relation holds, the distances between the lens and the mirrors become simply R_1 and R_2. Notice that the configuration shown in Fig. 2(d) reduces to the configuration shown in Fig. 1(b) when R_2 tends to infinity, if we keep the upper signs.

C. Periodic Lens and Lenslike Medium Degenerate Cavities

The case of M identical lenses of focal lengths f with equal optical separations c is easily treated with the help of the Sylvester's theorem.[10] The round-trip ray matrix is equal to $[-1]^\mu$ when

$$c/f = -2\cos(\mu\pi/M), \tag{13}$$

where μ is an integer equal to the number of (double) foci encountered by a ray pencil in a round trip.

A similar condition is obtained for a uniform lenslike medium with a refractive index $n = n_0 - \frac{1}{2}n_2x^2$, where x is a coordinate perpendicular to the optical axis,

$$L(n_2/n_0)^{\frac{1}{2}} = \mu\pi, \tag{14}$$

where L is the total path length. Equation (14) can be derived from Eq. (13) by letting μ/M tend to zero, and identifying the focusing properties of the two guiding media.

IV. Nonplanar Rings

By using again Eq. (8) we find that a ray matrix equal to $-[1]$ is obtained with either two confocal elements of equal focal lengths, or with three optical elements with optical separations opposite to the values given by Eq. (9). If the lenses are stigmatic, the ray matrix is equal to $-[1]$ in any meridional plane, and the transformation can be interpreted as an image rotation of π about the optical axis. Note that the position of the plane mirrors along the path is unimportant for the ray transformation and that these mirrors may be taken to be located at the same point on the path. This conclusion applies also to astigmatic elements if the orientation of these elements is properly modified. Accordingly, the plane mirror system must also provide an image rotation of π, for the total rotation to be 0 (mod 2π). This requirement is expressed in the next section from two points of view. First we consider the transformation resulting from given reflecting planes. Then we calculate the rotation experienced by off-set rays about a given optical axis.

A. Imaging by an Even Number of Plane Mirrors

The product of symmetries with respect to two planes is a rotation about their intersection equal to twice the angle that they make. The product of an arbitrary number of rotations is known to be the product of a single rotation and a translation along the rotation axis.[11] Clearly, a ray launched along such a rotation axis, in general, follows a closed path with a length L equal to the translation. The position of each straight section of the optical axis is obtained by considering the sequence of reflections which begins at the end of the section considered. This method neglects, however, possible interferences between the mirrors.

Let us consider more specifically four plane mirrors (P_1), (P_2), (P_3), (P_4), and call θ_1, θ_3 the angles between (P_1), (P_2) and (P_3), (P_4), respectively, ν the angle between the intersections of (P_1), (P_2) and (P_3), (P_4), and take the distance between these intersections as unity. By taking the product of the two rotations equivalent to (P_1), (P_2) and to $P(_3)$, (P_4), respectively, we get a round trip rotation Ω given by

$$\cos(\Omega/2) = \cos\theta_1 \cos\theta_3 - \cos\nu \sin\theta_1 \sin\theta_3. \tag{15}$$

This rotation is equal to π when

$$\cos\nu = \cot\theta_1 \cot\theta_3, \tag{16}$$

and the path length L is then given by

$$L^2 = -4\cos(\theta_1 + \theta_3)\cos(\theta_1 - \theta_3). \tag{17}$$

Let us apply these expressions to the case of two 90° roof tops with 90° between the edges (this is a combination of prisms which is often used in binoculars). Then, $\theta_1 = \theta_3 = \nu = \pi/2$. From Eqs. (15) and (17) we see that $\Omega = \pi$ and $L = 2$. The closed path coincides with the normal to the prism edges and a linear degenerate cavity is formed by introducing a self-confocal lens at the middle point between the prisms as shown in Fig. 3(a).

A similar result is obtained in the case of a corner cube and an arbitrary plane mirror as shown in Fig. 3(b). Taking the four mirrors by pairs we see that: $\theta_1 = \nu = \pi/2$, which again implies, from Eq. (15), that $\Omega = \pi$. The closed path coincides with the normal to the auxiliary plane mirror going through the corner cube top. These conclusions can be reached more readily by noting that a corner cube provides an inversion $(\mathbf{r} \rightarrow -\mathbf{r})$ with respect to its top, and by combining this inversion with the symmetry with respect to the auxiliary plane mirror.

Although Eqs. (17) and (16) easily give the total path length and the condition that the total rotation equal π, the calculation of the actual path of the optical

* This is not possible, however, in the special case in which $f_1 = f_2 = f_3 = f_4 \equiv f$ since, then, L must be equal to $8f$.

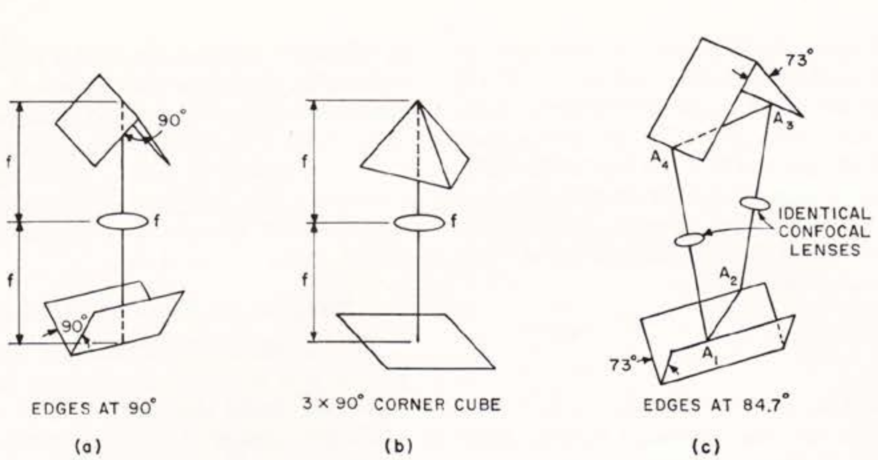

Fig. 3. These figures represent three nonplanar degenerate cavities. In (a) and (b), linear type cavities. In (c), ring type cavity.

axis is, in general, more intricate. In the next paragraph we instead choose a closed path with a free parameter and calculate the value required of that parameter for the rotation to be equal to π.

Let us consider a closed path $A_1A_2 \ldots A_{2N}A_1$ reflected from an even number of plane mirrors. The total rotation about the optical axis is[12]

$$\Omega = \beta_{12} - \beta_{23} + \beta_{34} \ldots - \beta_{2N\,1}, \tag{18}$$

where $\beta_{i\ i+1}$ is the angle between the incidence planes at A_i and A_{i+1}.

Let us consider as an example a closed optical axis having the shape of a twisted square $A_1A_2A_3A_4A_1$, where A_1, A_2, A_3, A_4 have, respectively, the following cartesian coordinates: $(1, -\tan\alpha, -1)$, $(-1, \tan\alpha, -1)$, $(-1, -\tan\alpha, 1)$, $(1, \tan\alpha, 1)$. When $\tan\alpha = (2^{-\frac{1}{2}} - 1/2)^{\frac{1}{2}}$ or $\alpha \simeq 24°5$, the angles between adjacent incidence planes are found to be equal to $\cos^{-1}[1/(1 + 2\tan^2\alpha)] = \pm\pi/4$, the plus and minus signs being alternately applicable. Consequently, the round trip rotation for this value of α is, from Eq. (18), $\Omega = \pi$. We also have: $\overline{A_1A_2} = \overline{A_2A_3} = \overline{A_3A_4} = \overline{A_4A_1} = 2/\cos\alpha = [2 + 2(2)^{\frac{1}{2}}]^{\frac{1}{2}} \simeq 2.2$.

The following properties of this path are easily obtained.

(1) The angles between $\overline{A_1A_2}$, $\overline{A_4A_3}$ and between $\overline{A_2A_3}$, $\overline{A_1A_4}$ are both equal to $2\alpha \simeq 49°$.

(2) The total path length is $L = 4[2 + 2(2)^{\frac{1}{2}}]^{\frac{1}{2}} \simeq 8.8$.

(3) The mirrors at A_1, A_2 and at A_3, A_4 form two roof tops with the same angle: $\theta = \cos^{-1}[2\sin^2\alpha/(1 + \sin^2\alpha)] = \cos^{-1}(1 - 2^{-\frac{1}{2}}) \simeq 73°$. The edges of these two roof tops make an angle: $\nu = \cos^{-1}[(1 - 4\tan^2\alpha)/(1 + 4\tan^2\alpha)] = \cos^{-1}\{[3 - 2(2)^{\frac{1}{2}}]/[2(2)^{\frac{1}{2}} - 1]\} \simeq 84°7$, and the distance between the edges is $4/\cos^2\alpha \simeq 4.8$. Notice that the introduction of the above values in Eqs. (15) and (17) gives again: $\Omega = \pi$ and $L = 4[2 + 2(2)^{\frac{1}{2}}]^{\frac{1}{2}}$.

From the discussion made before, this closed path constitutes a degenerate cavity if it incorporates two identical confocal lenses as shown in Fig. 3(c). We recall that a plane ring cavity requires at least three lenses to be degenerate.

B. Polarization

Let us give the value of the polarization matrix eigenvalues introduced in Sec. II. F. Upon reflection on a perfectly conductive plane mirror, the polarization vector experiences a symmetry with respect to the normal to the mirror. This symmetry is equivalent to the product of a symmetry with respect to the mirror plane and an inversion ($\mathbf{E} \rightarrow -\mathbf{E}$) with respect to the incidence point. The inversion (which is known to commute with symmetries) cancels out when the number of mirrors is even and the transformation is, in that case, the same as for an image, *viz.* a rotation equal to either 0 or π (mod 2π), depending on the type of degenerate cavity we are considering. In both cases, the polarization is degenerate and the polarization matrix eigenvalues are, respectively, $\psi = +1$ and $\psi = -1$.

To conclude this chapter, let us remark that the round trip ray matrices relative to an odd number of reflections are, respectively, equal to $+[1]$ and $-[1]$ in two perpendicular meridional planes. A ring cavity incorporating an odd number of mirrors consequently require, to be degenerate, astigmatic focusing elements.

V. Maxwell Fish-Eye and Luneburg Lens Cavities

We have discussed so far the first order properties of degenerate cavities incorporating lenses. It is of interest to point out that the Maxwell fish-eye, known since 1854, is by itself a degenerate cavity free of geometrical optics aberrations. The Maxwell fish-eye is a medium where the refractive index n varies as a function of the distance r to the origin as $n = n_0/(1 + r^2)$, where n_0 is the refractive index at the origin. It is known to image sharply any point in space. Since every ray trajectory is a circle[13] of optical legnth πn_0, it also constitutes a degenerate ring type cavity as defined in the introduction. Such a configuration is shown in Fig. 4(a), where the ray $r = 1$ is taken as the optical axis. The coupling with a source and a detector can be provided by a narrow slit in the medium acting as a beam splitter of low reflectivity. A degenerate cavity is also obtained if the Maxwell fish-eye medium

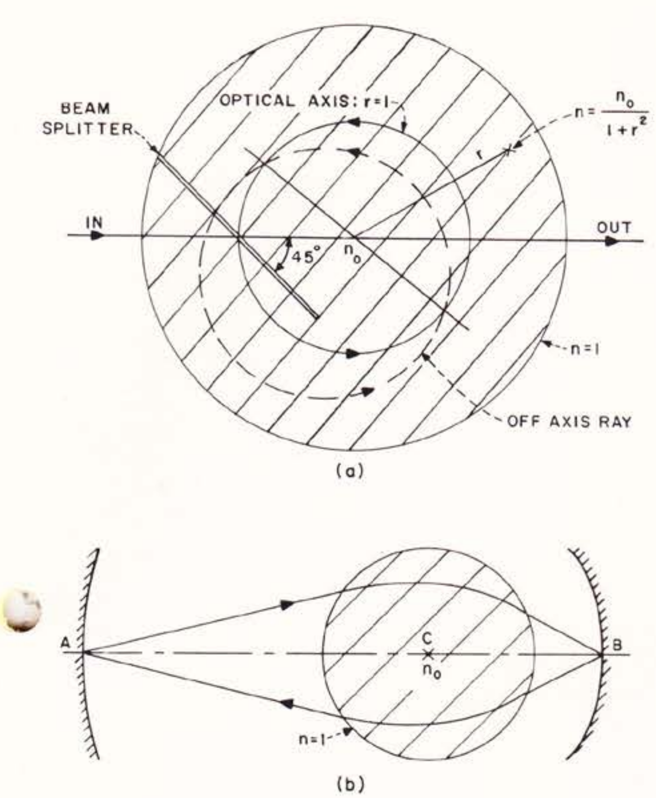

Fig. 4. (a) Maxwell fish-eye cavity. From a geometrical optics point of view, this cavity is rigorously degenerate; (b) rigorously degenerate cavity using the properties of a Luneburg lens.

is limited to the interior of a concentric reflecting sphere of radius unity. More flexible configurations can be obtained by making use of a generalized Luneburg lens,[14] which images sharply a point A into a point B in line with the lens center C. If two spherical mirrors with their centers at C are placed at A and B, respectively, as shown in Fig. 4(b), it is clear that any ray intersecting the lens retraces exactly its path after a round trip.

At optical wavelengths, these two configurations are of little practical significance, and the usual methods of lens correction have to be considered.

VI. Conclusion

Experiments* made on an active ring type cavity, which was degenerate in the plane of the ring and half-degenerate in the perpendicular direction, were found to be in agreement with the basic results of Sec. II. The degeneracy of the cavity was ascertained by observing that the response was not degraded by a moderate off-set of the incident beam in the ring plane. The acceptable off-set was limited, however, to ten beam waist radii because of the transverse variation of the laser gain. It was also observed that the transverse position of the cavity lenses was very critical in the ring plane (one-tenth of beam waist radius) and uncritical in the perpendicular direction, as one may expect from the discussion of Sec. II. D. The exact calculation of the field of view of a degenerate cavity requires further studies. When a degenerate cavity is misaligned and aberrated, an incident gaussian beam does not recover exactly its original shape and phase after a round trip. In addition, its axis is slightly off-set and tilted. As a consequence, the response of the cavity is shifted in frequency and the maximum output power reduced. These effects can be calculated, in principle, from the knowledge of the transformation of gaussian beams by refracting surfaces and the complex coupling coefficient between gaussian beams.

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* More detailed experimental results are given in Ref. 15.

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