Circuit theory of laser diode modulation and noise

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Abstract: The circuit theory of laser diode modulation and noise is based only on the energy and electron-number conservation laws, and on the well known expression $h\nu(G_b + G_a)$ for the spectral density of Nyquist noise currents. The conductances G_b and G_a represent stimulated absorption and stimulated emission, respectively. This theory leads to results that are in exact agreement with the predictions of quantum optics, even in the case of electronic feedback and non-classical states of light, but the optical field is not quantised. The theory is presented in a general form, applicable to arbitrary optical and electrical configurations, but is exemplified for only a single active element model of laser diode. For independent electron-hole injection, the results are the same as those obtained from standard rate equations, except for a phase-noise term. Important differences do occur for more realistic laser models.

1 Introduction

Oscillators modulated in power or phase can be used to transmit information, but the transmission is degraded by random fluctuations. The purpose of any laser theory is to express measurable quantities (for example, power fluctuations), in terms of parameters that pertain to the semiconductor material used in the active layer, to the geometric configuration of the laser diode, and to externally controlled parameters such as the injected electrical current. The modulation and noise properties of the laser at baseband frequency f that we wish to evaluate are:

(i) the modulation of the optical power by an injected current modulation,

(ii) the modulation of the optical phase by an injected current modulation,

(iii) random fluctuations of the electrical voltage across the diode,

(iv) random fluctuations of the optical power, (Note that the output current of an ideal detector reflects exactly the incident optical-power fluctuations as we define them. No electronic shot noise should be added.)

(v) random optical-phase fluctuations $\delta\phi(t)$, described by a spectral density $S_{\delta\phi}(f)$. The laser linewidth (full-width at half-power points) is the limit of $2\pi f^2 S$ when the baseband frequency f tends to zero,

(vi) correlations between the electrical voltage and the optical power and phase fluctuations.

Useful expressions concerning the laser-diode modulation and noise properties have been derived in the past from rate equations. These equations describe the time-evolution of the number of electrons in the active material and the number of photons in the optical cavity. For an exposition of the standard rate equations (SRE) the reader is referred to textbooks such as References 1–3.

In our opinion, the SRE are unsatisfactory both for conceptual and practical reasons [4]. First, the concept of the number of photons in the cavity should be avoided, because this quantity is not well defined in a medium with gain or loss. Secondly, the Langevin terms in the SRE can be justified only from heuristic arguments or complicated quantum-optics calculations. In fact, SRE may lead to totally incorrect results when the cavity conductance depends on frequency [5]. Attempts to apply SRE to multielement oscillator fluctuations so far led to unwieldy expressions.

The circuit theory was first proposed in an approximate form in Reference 6, and in an exact form in References 7 and 8. Our theory is capable of predicting the modulation and noise properties of multielement laser diodes on the basis of two simple concepts. The first is the law of electron-number and energy conservation. Let ν_0 denote the optical-cavity resonating frequency. Electrons are usually injected in the conduction band at an energy E_c such that $E_c - E_v > h\nu_0$, where E_v denotes the energy in the valence band corresponding to the same electronic momentum as E_c. Electrons cascade down in energy with the help of acoustical waves until the condition $E_c - E_v = h\nu_0$ is fulfilled. (Some energy is lost in that process but the number of electrons is preserved). The optical field existing in the cavity may then induce an electronic transition to the valence band. It collects the energy $h\nu_0$ lost by the electron. Note that we are, in fact, considering rates of electron generation or recombination and optical powers, rather than energy. Unlike the energy, the optical power is well defined in a medium with gain or loss. This process of stimulated emission is of major interest. Some of the electrons, however, reach the valence band spontaneously and deliver their energy to optical waves in other modes and at other frequencies (spontaneous radiative emission) or to other electrons (Auger effect). Or else, these electrons accumulate in the active material. The above concepts, when written in an appropriate mathematical form, suffice to establish the modulation properties of multielement laser diodes.

The second concept is that of amplitude and phase fluctuations of the optical field, which are obtained by giving consideration to the Nyquist noise currents that quantum mechanics associates with optical conductances.

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The double-sided spectral density of these currents is $h\nu_0(G_b + G_a)$, where the conductances G_b and G_a represent stimulated absorption and stimulated emission, respectively. Note that these noise currents are not given for free, so-to-speak: they play an essential role in the energy conservation law.

The concepts just discussed are well known in physics, but to our knowledge they have not been used jointly before. Nilsson and others [9] have proposed a circuit theory based on the second concept described above, but energy conservation was not enforced. As a result, the theory predicts correctly the laser linewidth but not the amplitude fluctuations. Conversely, the ad hoc assumption of a modified Nyquist current may lead to a correct expression for the amplitude but the expression for the phase is then invalid.

In order to implement the above principles, lumped-circuit elements are considered, whose dimensions are much smaller than the wavelength (hence the denomination, 'circuit theory'). For visible or near-infrared laser diodes, typical dimensions of the active region are $L = 250\ \mu m$, $w = 2\ \mu m$, and $d = 0.2\ \mu m$. Thus L and w are much larger than the operating free space wavelength, and the active region should be decomposed into smaller volumes. The general multielement formalism given in this paper (which is applicable to arbitrary optical and electrical configurations at any baseband frequency) is required in principle to account for the laser diode spatial inhomogeneities. However, the theory will be exemplified here only for a single active element, and compared with SRE. Some of the intermediate quantities introduced, such as the optical current or the conductances, are unusual in laser theory, but the final expressions are written in standard notation. Some changes of notation from Reference 8 have been made.

A number of important effects, such as the stability of the main-mode, damping of the relaxation-oscillation stronger than expected, nonthermal frequency deviations resulting from slow-current variations, or the influence of side-modes on the main mode linewidth, cannot be explained by the simple theory that assumes spatial homogeneity and linear gain. An explicit dependence of the optical conductance or gain on the optical field strengths (nonlinear gain) has been invoked to explain these effects. However, it may be that linear gain suffices to explain the observed facts if spatial inhomogeneities (along the three space directions) are fully accounted for. The circuit theory sketched at the end of the paper could nevertheless account for nonlinear gain if this effect turned out to be required. Thermal fluctuations and $1/f$ noise are not important above 1 MHz, and are not considered.

The simple-oscillator model discussed in detail in this paper seems to be approximately applicable to strongly index-guided spatially-homogeneous solitary diodes when the drive current exhibits shot-noise fluctuations. For most lasers, however, this simple model is not applicable, and the circuit theory leads to results that may differ drastically from those obtained from S.R.E.

Sinusoidal variations are denoted using an $\exp(-i2\pi\nu t)$ convention at optical frequencies, an $\exp(j2\pi ft)$ convention at baseband frequencies, and root-mean-square values are implied. f is assumed to be much smaller than ν. As far as random processes at baseband frequency f are concerned, double-sided spectral densities (Fourier-transforms of the covariances) are used. Note that if $A = \Sigma a_k x_k$, $B = \Sigma b_k x_k$ are the weighed sums of independent processes $x_k(t)$ of spectral density S_{x_k}, the cross-spectral density of A and B is

$$S_{AB} = \sum a_k^* b_k S_{x_k} \tag{1}$$

where the star means that j should be changed to $-j$. We call the normalised modulus of S_{AB} correlation.

2 Laser parameters

Let us consider first a small cylindrical direct band-gap semiconductor sample, of cross-section area Lw and thickness d. This piece of semiconductor material exhibits an optical conductance $G(N)$, where N is the number of electrons in the conduction band. It is connected to an external cavity modelled as a parallel susceptance $B(\nu)$, as shown in Fig. 1, which vanishes at the resonant frequency ν_0 (the label '1' is omitted in the present section). The laser (semiconductor element plus cavity) is loaded by a detector of positive conductance G_2. The transmission line shown in the figure connecting the laser to the detector may be realistic at far-infrared frequencies, but at optical frequencies it should be considered as modelling the appropriate optics (focussing lenses). Self-oscillation occurs when the semiconductor optical conductivity is negative and attains a sufficiently large absolute value:

$$G(N) \approx -G_2 \tag{2}$$

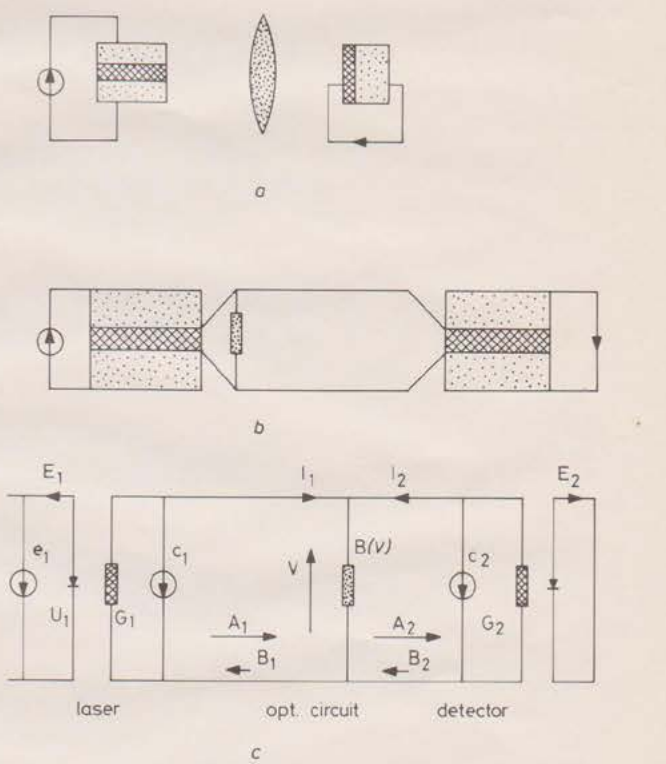

Fig. 1
a: the laser diode output is focussed on a detector
b: schematic representation of the optical cavity, modelled as a parallel capacitance-inductance circuit, shown separated from the negative conductance semiconductor, and focussing lens is represented by a transmission line
c: circuit representation of negative-conductance laser diode (label '1') and positive-conductance detector diode (label '2'). The optical cavity is represented by the parallel susceptance $B(\nu)$, labelled 'optical circuit'. The laser diode is current driven with electronic rate E_1 and noise current e_1. I_1, I_2 are optical currents and V is the optical voltage across the circuit. c_1, c_2 are Nyquist noise currents. The A-waves are forward propagating waves, the B-waves are weak backward propagating waves

A semiconductor is characterised primarily by its bandgap energy E_g corresponding to a bandgap optical frequency ν_g given by $h\nu_g = E_g$. When the semiconductor

is submitted to an optical field at frequency ν, the perturbation of the electronic wave functions results in an induced optical current that can be expressed by a conductivity σ and a permittivity ε. For a cylindrical piece of semiconductor the admittance is

$$Y(\nu, N) \equiv G(\nu, N) + iB(\nu, N)$$
$$= (\sigma - i2\pi\nu\varepsilon)Lw/d \tag{3}$$

According to simple theories, neglecting band-gap shrinkage and band-tail states, the minimum value of $G(\nu)$ (maximum gain) for some N-value occurs at a frequency ν exceeding slightly ν_g. For the single-mode operation considered in this paper, the weak dependence of Y on ν can be neglected, and for simplicity, we also assume that the steady state value $B(\nu_0, N_0) = 0$.

In laser diodes, the electrons are injected from an electrical power supply into the conduction band of the semiconductor with the help of a PIN junction. As we discussed in the introduction, most of these electrons deliver their energy to the oscillating optical mode as they fall down into the valence band, but some of them deliver their energy to nonoscillating modes, mainly radiation modes, or to other electrons or to acoustical waves (nonradiative recombination). Otherwise, they accumulate in the active region. The law of conservation of electron number and energy therefore reads

$$\left(E + S + \frac{dN}{dt}\right)h\nu_0 + \text{opt. power} = 0 \tag{4}$$

In eqn. 4, E is the rate at which electrons are injected into the semiconductor, outgoing electron rates being defined as positive for later convenience. The rate S at which electrons reach the valence band spontaneously (with an energy loss which need not be specified here), is a function of N. dN/dt (if positive) is the rate at which electrons accumulate. The term in parenthesis in eqn. 4 is thus the number of electrons that deliver an energy $h\nu_0$ to the oscillating mode, per unit time. We find it convenient to denote the outgoing optical power in the oscillating mode as $Ph\nu_0$. However, it would be misleading from a conceptual point of view to think of P as being a photon generation rate. The concept of photons (elementary excitations of the quantised optical field) is not used in this paper.

A relation analogous to eqn. 4 applies to detectors as well, but in that case E is positive, P is negative, and S can be neglected because the carrier density is small. If the output P from a laser diode is focussed on an ideal detector, the electronic rate E at the detector output is equal to P at small frequencies f. This conclusion, again, does not imply that the incident light consists of photons, but only that energy and electron number are conserved quantities.

Let us first consider time-averaged values. Eqn. 4 and the oscillation condition given earlier read, respectively,

$$E + S(N) + P = 0 \tag{5a}$$

$$G(N) \approx -G_2 \tag{5b}$$

E may be specified if an infinite internal impedance electrical source is used to drive the diode. Otherwise, E depends on the electrical voltage U across the laser diode through Ohm's law. The electrical voltage U across the diode is equal to the energy spacing between quasi-Fermi levels in the valence and the conduction bands, divided by q, if one neglects the voltage drop in the confining layers and contacts. At a given temperature T, the electronic density in the conduction band (equal to the hole density in the valence band for undoped active layers) is a unique function of U. The voltage U is a monotonically increasing function of the number N of electrons in the conduction band, $U = U(N)$. For a laser diode, $U > h\nu$, while $U < h\nu$ for a detector.

As far as the spontaneous emission rate S is concerned, let us note that it takes, on average, a time of τ_s (approximately the same for all the electrons in the band) for an electron in the conduction band to fall spontaneously into an empty state of the valence band. Thus, if the k-conservation rule holds, the spontaneous recombination rate S at $T = 0K$ is equal to N/τ_s. There are deviations from this linear law, however, resulting from nonzero temperature, relaxation of the k-conservation rule (in which case S may be proportional to N^2), or nonradiative Auger recombination ($S \propto N^3$), and we set the spontaneous recombination rate S as some function $S(N)$, in general.

Eqn. 5b can be solved for N and the result carried into eqn. 5a. In conventional notations, the time-average of the quantities S, E and P are written as

$$S_0 \equiv I_{th}/q$$
$$-E_0 \equiv I_e/q$$
$$P_0 \equiv \text{av. opt. power}/h\nu_0 \tag{6}$$

where I_e is the injected current, I_{th} the threshold current, q the absolute value of the electron charge, and $h\nu_0$ the transition energy. From eqns. 5 and 6, we obtain the well known relation

$$\frac{\text{av. opt. power}}{h\nu_0} = \frac{I_e - I_{th}}{q} \tag{7}$$

In summary, theoretical analysis or measurement of the electronic and optical properties of the semiconductor sample provides three functions: $U(N)$, $S(N)$ and $Y(N)$. The average oscillation parameters, ν_0, N_0, U_0, P_0, follow from the oscillation condition and the time-average of eqn. 4.

In order to evaluate the diode modulation and noise one needs to know how E, S, P deviate from their average values. The first-order variation of the conservation law in eqn. 4 reads at baseband frequency f for any element,

$$(j2\pi f + S_N)\delta N + \delta E + \delta P + s_n = 0 \tag{8}$$

where we set

$$\delta E = E_N \delta N + e_n + m \tag{9a}$$

$$\delta S = S_N \delta N + s_n \tag{9b}$$

$$\delta P = P_N \delta N + p_n \tag{9c}$$

where δN is the variation of the electron number N and subscripts indicate derivatives, e.g.: $E_N \equiv dE/dN$. The terms e_n, s_n, p_n are additive noises, and m expresses the injected current modulation.

Let us show how the terms in eqns. 8 and 9 can be evaluated. The parameter S_N follows from the semiconductor properties as we have seen earlier. Because spontaneous emission is supposed to consist of independent events, the spectral density of s_n is given by the usual shot-noise formula

$$S_{s_n} = S(N) \tag{10}$$

Consider next δE. If the active elements are interconnected electrically, eqn. 9a should be written in matrix notation

$$\delta E = E_N \delta N + e_n + m \quad (11)$$

In the absence of electronic feedback and carrier diffusion between active elements, E_N is a diagonal matrix. Its elements depend on the driving-circuit admittances $Y_e(f)$. Leaving aside noise terms and modulation we have, using Ohm's law,

$$\delta E = -\delta I_e/q;\ \delta I_e = Y_e \delta U = Y_e U_N \delta N \quad (12)$$

Therefore,

$$E_N \equiv \delta E/\delta N = -Y_e U_N/q \quad (13)$$

For simplicity we now assume that the driving circuit matrix admittances vanish $E_N = 0$. The spectral densities of the noise terms e_n depend in general on the driving circuitry. If the driving circuit generates shot-noise, the spectral density of e_n is equal to $|E|$. In general, we set the spectral density of e_n as equal to $\xi|E|$, $\xi \geqslant 0$, and the cross-spectral densities are supposed to vanish.

The noise term p_n has its origin in the complex Nyquist-like noise current $c(t) \equiv c'(t) + ic''(t)$ associated with conductances according to the fluctuation-dissipation theorem. The conductance G is written as $G_b - G_a$, where the conductance G_b is the result of stimulated absorption, and the conductance G_a to stimulated emission. The (double-sided) spectral densities of the Nyquist currents are [14]

$$S_{c'} = S_{c''} = h\nu(G_b + G_a) = h\nu G(1 - 2n_s) \quad S_{c'c''} = 0 \quad (14)$$

where the so-called spontaneous emission factor

$$n_s \equiv \frac{G_a}{G_a - G_b} \quad (15)$$

is unity for complete population inversion (strong pumping at $T = 0K$), and of the order of 2 at room temperature.

To summarise, the semiconductor material is characterised by five dimensionless parameters:

$$u \equiv (N/U)U_N \quad s \equiv (N/S)S_N \quad g \equiv (N/G)G_N$$
$$n_s \quad \alpha \equiv -B_N/G_N \quad (16)$$

The phase-amplitude coupling factor α is of the order 5. These parameters are constant above threshold because we assume that N is clamped to its threshold value.

Let us now consider the laser geometry. As shown in Fig. 1, a simple model of optical cavity is a parallel susceptance $B(\nu)$. If G_2 denotes the load conductance, the cold cavity linewidth f_0 is defined from

$$1/2\pi f_0 \equiv \tau_p = -B_\nu/4\pi G_2 \quad (17)$$

If B consists of a capacitance C, and an inductance L in parallel, then for example

$$B(\nu) = -2\pi\nu C + 1/2\pi\nu L \approx -4\pi C\delta\nu$$
$$\delta\nu \equiv \nu - \nu_0 \quad LC(2\pi\nu_0)^2 = 1$$
$$\tau_p \equiv 1/2\pi f_0 = C/G_2 \quad (18)$$

and for a laser with length L, group velocity v_{gz}, internal power loss α_{pz}, and mirror power reflectivities R_1, R_2, we have approximately

$$2\pi f_0 = v_{gz}\left(\alpha_{pz} + \frac{1}{2L}\ln\frac{1}{R_1 R_2}\right) \quad (19)$$

Finally, we shall make use of four operating parameters:

$$\zeta \equiv S/P \quad r \equiv \tau_s/\tau_p = 2\pi f_0 N/S$$
$$x \equiv f/f_0 \quad \xi(x, \zeta) \quad (20)$$

ξ expresses the injected current fluctuations. It may depend on f and on the bias injected current. A standard value is $\xi = 1$ (shot-noise).

All the laser diode properties can be expressed in terms of the nine dimensionless quantities: u, s, g, α, n_s, ζ, r, x and ξ. Because some of these quantities enter in group, it is convenient to define the following

$$\eta \equiv 1/(1 + \zeta)$$
$$C \equiv (\xi + 1)\zeta + \xi$$
$$D \equiv g + s\zeta + j(r\zeta x - g/x)$$
$$F \equiv [C - 1 + 2n_s(1 + 1/x^2)]/|D|^2 \quad (21)$$

In numerical applications, we shall use the parameter values in Table 1 below that are adapted from Table 6.1 and 6.2 of Reference 2. dU/dN, not given in Reference 2, has been evaluated separately. These numerical values are applicable to buried heterostructures operating at a free-space wavelength of 1.3 μm, with geometrical dimensions $L = 250$ μm, $w = 2$ μm, $d = 0.2$ μm. $h\nu_0 P$ is the optical power in the oscillating mode, dissipated internally in the laser diode as well as externally. The S-value given corresponds to a threshold current $qS = 15$ mA, and the operating current $I_e = q(S + P) = 77$ mA for the P-value indicated.

Table 1: Numerical values of the parameters

$U = 1$ V	$N = 2.1\ 10^8$	$S = 9.6\ 10^{16}\ s^{-1}$
$u = 0.1$	$s = 1.8$	$g = 1.9$
$n_s = 2$	$\alpha = 5$	$f_0 = 100$ GHz
$P = 4\ 10^{17}\ s^{-1}$ (61 mW)		
$\zeta = S/P = 0.24$, $r = 2\pi f_0\ N/S = 1\ 370$.		
$\xi = 0$ or $\xi = 1$.		

3 Evaluation of $\delta P = P_N \delta N + p_n$

The purpose of the present section is to evaluate the variation δP of the normalised optical power P, which enters into the conservation law, eqn. 8. This quantity splits into a term proportional to the carrier number variation δN, and a noise term that we denoted p_n.

Let us consider a conservative (lossless-gainless) linear n-port optical circuit. Nonlinear semiconducting elements of steady-state optical conductances G_{0k}, are connected at the ports. Subscripts '0' refer to steady-state values and subscripts $k = 1, \ldots n$ label the ports. These subscripts are omitted when no confusion may arise. The diode series resistance, stray capacitance and lead inductance are parts of the electrical circuit, while the optical cavities and the steady-state optical susceptances belong to the n-port optical circuit. Let V denote the optical voltage across the active element, and I the optical current entering into the optical circuit (see Fig. 1). We define A and B-waves [not to be confused with susceptances denoted $B(\nu)$] according to

$$A \equiv G_0 V - I \quad B \equiv G_0 V + I \quad G_0 V_0 + I_0 = 0 \quad (22)$$

Note that the steady-state conductances G_0 can be either positive or negative. It follows from the expression $Re(V^*I)$ of the power flow that A-waves go from negative conductance elements to the optical circuit or from the optical circuit to a positive conductance element, while B waves are weak counterpropagating waves that vanish in the absence of perturbations. The steady-state optical power going from the optical circuit to a nonlinear element is $h\nu P = G_0|V_0|^2$.

Now consider small deviations denoted by δ: $\delta V \equiv V - V_0$, $\delta I \equiv I - I_0$, $\delta G \equiv G - G_0$, $\delta A \equiv A - A_0$, $\delta B \equiv B$.

Eqn. 22 gives to the first order

$$\delta A = G_0 \delta V - \delta I \quad B = G_0 \delta V + \delta I \quad (23)$$

Kirchhoff's law reads (see Fig. 1)

$$I + c + YV = 0 \quad Y = G_0 + \delta G(1 - i\alpha) \quad (24)$$

where $c \equiv c' + ic''$ denotes the Nyquist-like noise current and α the phase-amplitude factor defined earlier. Using the definition of B in eqn. 23, the first-order variation of eqn. 24 is

$$B + \delta G(1 - i\alpha)V + c = 0 \quad \delta G = G_N \delta N \quad (25)$$

This relation shows that the counter-propagating B-waves consist of a term proportional to δN and the Nyquist noise source. For a linear element (e.g. an ideal detector), $\delta G = 0$ and only the Nyquist source needs to be considered.

We are now in position to evaluate the first-order variation of the optical power

$$h\nu P = \mathrm{Re}\,(V^* I) \quad (26)$$

Using the expression in eqn. 23 of δA we obtain

$$\delta P = \mathrm{Re}\,\{(V^*/h\nu)\delta A\} \Leftrightarrow \delta P/2P = \mathrm{Re}\,\{\delta A/A\} \quad (27)$$

Thus, the variation of the optical power emitted by an element with gain or absorbed by an element with loss, is proportional to the real part of the forward propagating A-wave variation. The backward-propagating B waves do not contribute directly to δP in a linearised theory ($\delta A \ll A$), but they are nevertheless essential to the formulation (see below). The preceeding relations apply to any of the nonlinear elements.

Let the n-port linear optical-circuit be characterised at some optical frequency ν by a matrix impedance $\boldsymbol{Z}(\nu)$

$$V = \boldsymbol{Z}(\nu)I \quad (28)$$

The propagating A and B-waves are related by a scattering-like matrix

$$B = \boldsymbol{S}(\nu)A$$
$$\boldsymbol{S} = \boldsymbol{1} + 2(\boldsymbol{G}_0\boldsymbol{Z} - \boldsymbol{1})^{-1} \quad (29)$$

where $\boldsymbol{G}_0$ is a diagonal matrix with elements G_{0k}, and $\boldsymbol{1}$ is the identity matrix. (Because we use unconventional notations for the A and B-waves, $\boldsymbol{S}$ is not symmetrical for a reciprocal circuit, and not unitary for a conservative circuit).

To treat small variations occurring at baseband frequency f, it is convenient to use the bi-complex notation discussed in Appendix A, in which 'i' and 'j' refer to time variations at optical and electrical frequencies, respectively. Re (.) and Im (.) refer to 'i' exclusively. The relation between the A and B-wave variations is

$$B = \boldsymbol{S}(\nu_0 + ijf)\delta A \approx (\boldsymbol{S}_0 + \boldsymbol{S}_\nu ijf)\delta A$$
$$\boldsymbol{S}_0 \equiv \boldsymbol{S}(\nu_0);\ \boldsymbol{S}_\nu \equiv d\boldsymbol{S}/d\nu \quad (30)$$

Because $f \ll \nu_0$, a first-order expansion of $\boldsymbol{S}$ is sufficient. In general, $\boldsymbol{S}_0$ is finite but singular. In the special case treated below one must invert directly $\boldsymbol{S}(\nu)$.

Finally, introducing eqns. 30 and 25 into eqn. 27, we obtain

$$\delta P_k = \mathrm{Re}\,\{(V_k^*/h\nu)\delta A_k\}$$
$$\delta A = [\boldsymbol{S}(\nu_0 + ijf)]^{-1}B$$
$$-B_m = G_{mN}(1 - i\alpha_m)V_m \delta N_m + c_m \quad (31)$$

δP_k is expressed in eqn. 31 as the sum of terms proportional to the δN_k and noise terms. Eqns. 8–15 and 31 solve the general problem.

4 Solution for two elements (laser + detector)

Our formalism is now applied on the 2-element configuration shown in Fig. 1. Subscripts 1 refer to the laser semiconducting element. Subscripts 2 refer to the detector. The detector is assumed to be linear, that is, to have a constant conductance. In that case it would be possible to treat the detector as part of the optical circuit and use an S-matrix of order 1: the reflexion coefficient. But because the optical circuit becomes nonconservative, there are noise waves generated by the optical circuit. Such a one-element theory is simple as far as modulations are concerned, but for random fluctuations the 2×2 formalism used below turns out to be more convenient.

The 2×2 $\boldsymbol{Z}$-matrix of the optical circuit is easily obtained from Ohm's law: $I_1 + I_2 = iB(\nu)V$, where $V \equiv V_1 = V_2$ taken as real for convenience denotes the optical voltage across the circuit. The parallel susceptance $B(\nu)$ is given in eqn. 18. The expression of the 2×2 scattering matrix $\boldsymbol{S}$ follows from eqn. 29. The relation between the counterpropagating B-waves and the fluctuations of the forward-propagating δA-waves

$$\delta A = \{\boldsymbol{S}(\nu_0 + ijf)\}^{-1}B \quad (32)$$

is explicitly

$$\delta A_1 = -(1 + z)B_1 - zB_2 \quad (33a)$$
$$\delta A_2 = zB_1 - (1 - z)B_2 \quad z \equiv f_0/ijf \equiv -j/x \quad (33b)$$

If we substitute eqn. 33 into eqn. 31, we obtain

$$\delta P_1 = (1 + z)gP\delta N/N + (V/h\nu)[(1 + z)c_1' + zc_2'] \quad (34a)$$
$$\delta P_2 = -zgP\delta N/N + (V/h\nu)[-zc_1' + (1 - z)c_2'] \quad (34b)$$

where we have replaced $-G_1 V^2/h\nu$ by the steady-state value P, set $\delta N_1 \equiv \delta N$, $\delta N_2 = 0$, and

$$g \equiv g_1 \equiv \frac{N}{G_1}\frac{dG_1}{dN} \quad (35)$$

Let us now substitute the expression of δP_1 from eqn. 34a into the particle-rate conservation law, eqn. 8, applied to the nonlinear element 1. We obtain

$$D_n \delta N + s_n + e_n + m + p_n = 0 \quad (36)$$

where

$$D_n \equiv j2\pi f + S_N + (1 + z)gP/N \quad (37)$$

Note that the laser-diode complex relaxation frequency f_{rc} is a root of $D_n(f)$. We also obtain

$$p_n \equiv (V/h\nu)[(1 + z)c_1' + zc_2'] \quad (38)$$

Eqn. 36 gives δN as a function of independent noise sources. The spectral densities of the Gaussian processes s_n, e_n, c_1', c_2' are, respectively, S, $\xi|E| \equiv \xi(S + P)$,

$h\nu(2n_s - 1)G_2$ and $h\nu G_2$. We have therefore

$$S_{p_n} = \{[1 + (f_0/f)^2](2n_s - 1) + (f_0/f)^2\}P \tag{39}$$

The formulae will now be expressed in terms of the dimensionless parameters defined in eqns. 16, 20 and 21.

5 Laser diode admittance

The diode admittance Y_{oe} ('oe' for optoelectrical) is the ratio of electrical current and voltage variations. Its expression follows from eqns. 36 and 37 with the noise terms suppressed.

$$Y_{oe} \equiv \frac{\delta I_e}{\delta U} = -qN_U \frac{m}{\delta N}$$
$$= qN_U[j2\pi f + S_N + (gP/N)(1 + f_0/jf)]$$
$$\equiv j2\pi f C_{oe} + G_{oe} + (j2\pi f L_{oe})^{-1} \tag{40a}$$

where we have introduced, as in References 11 and 12, equivalent electrical circuit elements Y_{oe} is modelled as a capacitance C_{oe}, a conductance G_{oe} and an inductance L_{oe} in parallel. For the parameter values in Table 1, we calculate $C_{oe} = 336$ pF, $G_{oe} = 1.4$ Siemens, $L_{oe} = 1.3$ pH. The complex relaxation frequency is $f_{rc} = 7.6 + j0.35$ GHz. As is well known, the relaxation frequency (resonance of C_{oe} and L_{oe}) increases as the square root of the emission rate P. As P increases, the conductance G_{oe} also increases and the relaxation gets more strongly damped. A complete model of diode admittance should include the series resistance $R_s \approx$ a few ohms, the lead inductance, and a parasitic capacitance. The non-reciprocity of the equivalent optoelectrical circuit appears only for a more complete detector model.

A convenient form is obtained by dividing Y_{oe} by the static diode admittance $Y_{oes} \equiv I_e/U$. We obtain

$$\frac{Y_{oe}}{Y_{oes}} = \eta D/u \tag{40b}$$

6 Electrical voltage fluctuation

The basic relation is again eqn. 37 in which we set now $m = 0$, but keep the noise terms

$$\delta N = -(e_n + s_n + p_n)/D_n \tag{41}$$

The spectral density of $\delta U \equiv U_N \delta N$ is given by

$$PS_{\delta U/U} = u^2 F \tag{42}$$

In the limit in which $f \to 0$, the spectral density of $\delta U/U$ is, more explicitly,

$$S_{\delta U/U} = \left(\frac{U}{G_1}\frac{dG_1}{dU}\right)^2 \frac{2n_s}{P} \tag{43}$$

At large power levels, eqn. 43 applies to any frequency [8]. For the parameter values in Table 1, we calculate $S_{\delta U/U} = 36\ 10^{-15}$ (Hz)$^{-1}$.

7 Optical power modulation

According to our sign convention, the variation δP of the normalised optical power emitted by the laser diode is equal to δP_2, and the injected electronic rate is $-m$. The expression for δP follows from eqn. 34b without the noise terms, and with $\delta N = -m/D_n$. In terms of the dimensionless quantities in eqns. 20 and 21:

$$\delta P/(-m) = g/jxD \tag{44}$$

The ratio in eqn. 44 is unity at small baseband frequencies when both electron and photon storage can be neglected. Indeed, in that limit, N does not vary and thus the rate of spontaneous emission S is invariant. Since the quantum efficiency is assumed to be unity, it follows that the normalised optical-power variation equals the electron-rate variation. In terms of optical power and electrical current we have more explicitly,

$$\frac{\text{opt. power mod.}}{\text{current mod.}} = \frac{h\nu}{q}[1 + jx(1 + s\zeta/g) - (r\zeta/g)x^2]^{-1} \tag{45}$$

8 Optical power fluctuation

The same basic relation in eqn. 34b is used, setting this time $m = 0$, but keeping the noise terms. We obtain:

$$4PS_{\delta P/2P} = 1 + [(C-1)g^2 + 2n_s\zeta^2(s^2 + r^2x^2)]/x^2|D|^2 \tag{46}$$

The variation of $S_{\delta P}$ is represented as a function of f in Fig. 2 for the parameter values in Table 1. Measurements do not in fact show relaxation peaks as pronounced as predicted by the simple theory. To explain the strong damping of the relaxation oscillations, nonlinear gain has been invoked. There are, however, other mechanisms that could be responsible: weak transverse guidance, transverse diffusion, and longitudinal carrier-induced gratings. $S_{\delta P}$ is also represented in Fig. 2 for the case in which the injected current does not fluctuate ($\xi = 0$). The difference is significant at low frequencies.

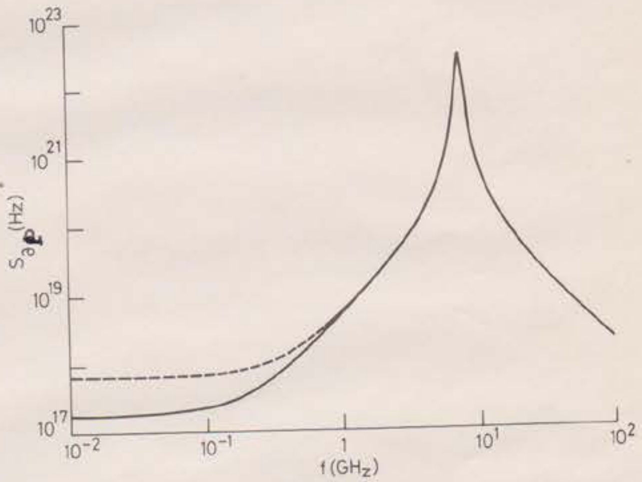

Fig. 2 *Variation of spectral density of optical power fluctuations, as function of baseband frequency f in GHz, with ($\xi = 1$) and without ($\xi = 0$) injected-current fluctuations*

——— no shot noise
- - - - shot noise

The correlation between the optical-voltage fluctuation and the optical-power fluctuation at small f-values is

$$C_{PU} = \left\{1 + \frac{Cg^2}{2n_s s^2\zeta^2}\right\}^{-1/2} \tag{47}$$

C_{PU} is shown in Fig. 3 as a function of $P/S = I_e/I_{th} - 1$. The correlation coefficient decreases from unity at threshold down to zero at large powers. At large powers, this correlation coefficient vanishes at every frequency [8]. The variation of C_{PU} for the case where the injected current does not fluctuate is also shown. We see that C_{PU} depends strongly on the ξ-value. For measurement purposes, $f = 10$ MHz would be an appropriate frequency. It approximates well the low-frequency limit of the theory, and is sufficiently high that thermal effects and $1/f$ noise should be negligible.

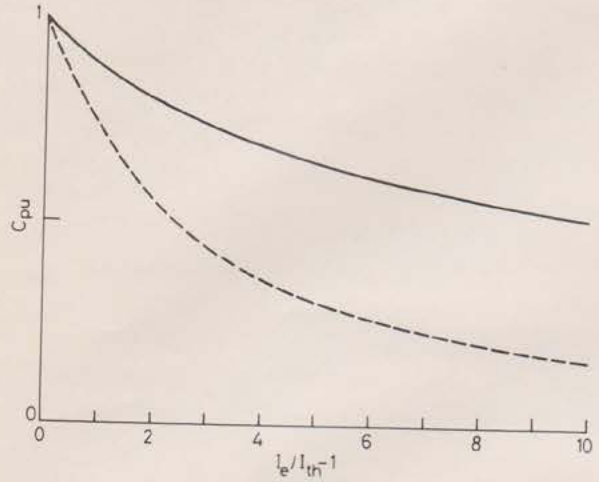

Fig. 3 *Correlation coefficient between electrical-voltage across the laser diode and the optical-power fluctuations at $f = 10$ MHz as function of the ratio of injected current and threshold current, with ($\xi = 1$) and without ($\xi = 0$) injected-current fluctuations*

——— no shot noise
- - - - shot noise

9 Phase modulation

The phase deviation ϕ follows from the concept of complex phase

$$\frac{1}{2}\frac{\delta P}{P} + i\delta\phi = \frac{\delta A}{A} \tag{48}$$

or more specifically, from analysis of the homodyne-detection process. While the optical power variation is related to the real part of $\delta A/A$, the phase variation $\delta\phi$ is related to the imaginary part of $\delta A/A$.

We first obtain from eqn. 31 and 33 at any frequency

$$\delta\phi = \delta\phi_2 = -\frac{1}{2}\alpha\frac{\delta P}{P} \tag{49}$$

where α is the phase-amplitude factor defined earlier and δP is given in eqn. 44. Unless α is very large, a phaseshift of π cannot be obtained by a small-signal current modulation according to the present theory. Eqn. 49 provides a means of measuring α. However, the predicted phase difference of π between $\delta\phi$ and δP is found experimentally at high frequencies only.

10 Phase fluctuation and linewidth

The spectral density of the phase fluctuation $\delta\phi \equiv \delta\phi_2$ is obtained from eqns. 48, 31 and 33, keeping the noise terms but suppressing the modulation. We obtain

$$4PS_{\delta\phi} = 1 + \frac{2n_s + \alpha^2g^2F}{x^2} \tag{50}$$

In the limit of large frequencies, eqns. 46 and 50 give

$$S_{\delta P}S_{\delta\phi} = P \times \frac{1}{4P} = \frac{1}{4} \tag{51}$$

(the right-hand-side of eqn. 51 is unity if single-sided spectral densities are used. This is twice the minimum value allowed by quantum optics for any state of light).

The variation of $S_{\delta\phi}$ is shown in Fig. 4 as a function of f. The effect of injected current fluctuations, while strictly nonzero, is too small to appear on the curves (less than 1%).

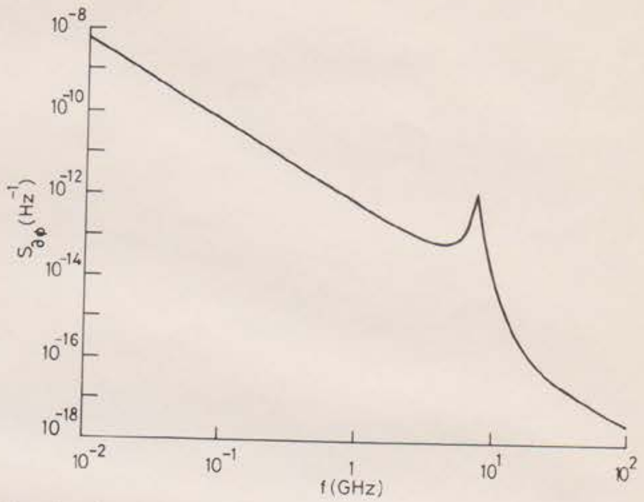

Fig. 4 *Spectral density in Hz^{-1} of the phase fluctuations as function of the baseband frequency f in GHz. The effect of injected current fluctuations (less than one per cent) is too small to appear on the curve*

The laser linewidth follows from the low-frequency behaviour of $S_{\delta\phi}$ as shown in the introduction. We obtain

$$2\pi\Delta\nu P/h\nu = (2\pi f_0)^2 \frac{1}{2}(1 + \alpha^2)n_s \tag{52}$$

for any ξ-value. The conclusion that the laser linewidth does not depend on injected current fluctuations holds only for a single active element.

The correlation between phase and amplitude fluctuations at low baseband frequencies is

$$C_{P\phi} = \frac{\alpha}{\sqrt{(1 + \alpha^2)}}C_{PU} \tag{53}$$

The factor C_{PU} given in eqn. 47 is unity, and can be omitted only just above threshold.

11 Conclusion

We have shown that the circuit theory of laser diode modulation and noise, which is based on familiar conservation laws (number of electrons and energy) and Nyquist currents, is capable of predicting all the quantities of interest at any baseband frequency and for arbitrary optical and electrical configurations. The theory has been exemplified only for a one-active element laser-diode model. In that case, and assuming shot-noise injected current fluctuations, we find an exact agreement with the standard rate equations (SRE), except for a phase term which must be added ad hoc to the SRE (see Appendix B). Yamamoto has pointed out that there is no particular reason why the current injected in a laser diode should exhibit shot-noise fluctuations. We have considered this situation only for the sake of comparison with SRE. In a real (multielement) laser, the precise value of the injected-current density fluctuations would be difficult to evaluate, as it depends on complicated electronic processes. Significant differences do occur when the injected current fluctuations are modified, e.g. suppressed. This point is of particular interest with respect to the low-frequency correlation between the optical power and the electrical voltage.

From a practical viewpoint, the major differences between the circuit theory and SRE occur when the cavity conductance depends on the optical frequency at the operating point, as is the case for lasers with gain-guidance or weak guidance. This dependance entails an enhanced relaxation oscillation damping, and a linewidth enhancement not given simply by the K-factor [5].

Only a single optical mode has been considered. It has been found experimentally that side-modes of moderate power may importantly enhance the main-mode linewidth. This experimental result cannot be explained by the simple theory that postulates linear gain and spatial homogeneity as was shown in Reference 15. An explanation of the observations based on nonlinear gain [3] is plausible. However, spatial variation of the α factor constitutes an alternative mechanism [4] that should be investigated in detail.

The circuit theory should be ultimately capable of treating laser diodes having arbitrary spatial inhomogeneities in three dimensions, with the help of a finite-element method. For that purpose, the S-matrix formulation given in this paper may not be the most practical. It is preferable to set the optical-current and voltage variations for some active element as $\delta I = I' + iI''$ and $\delta V \equiv V' + iV''$, respectively, and to write down the linear relationship existing between $[I'\ I'']$ and $[V'\ V'']$, with additional noise terms. For example, if the laser length is subdivided into $n = 1000$ elements, this alternative procedure requires only multiplication of n 4×4 matrices, while the S-matrix formalism requires inversion of a $n \times n$ matrix.

In conclusion, it appears that the circuit theory [8] is more directly tied up to admitted physical laws than SRE. It essentially agrees with SRE for simplified models but is more accurate in general situations. The theory appears to be in exact agreement with Quantum Optics, but is much simpler to use because the optical field is not quantized.

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13 Appendix A: the bicomplex notation

For brevity, let us set $\omega \equiv 2\pi\nu$, $\Omega \equiv 2\pi f$. Consider first the standard complex notation at optical frequencies. We associate with the complex number A a real function of time $A(t)$ according to

$$A(t) = \mathrm{Re}\,\{A \exp(-i\omega t)\} \tag{54a}$$

and the response of a linear system is

$$B(t) = \mathrm{Re}\,\{B \exp(-i\omega t)\} \quad B = S(\omega)A \tag{54b}$$

where $S(\omega) = S^*(-\omega)$ is the complex response function.

Now consider the signal

$$A(t) = (c' \cos \Omega t - c'' \sin \Omega t) \cos \omega t + (d' \cos \Omega t - d'' \sin \Omega t) \sin \omega t \tag{55}$$

which represents sinusoidal modulations at angular frequency Ω of both the in-phase and the quadrature components of the optical signal at angular frequency ω. The $A(t)$ spectrum contains only the frequencies $\omega + \Omega$ and $\omega - \Omega$ in the positive frequency domain.

Standard methods give the response

$$B(t) = \tfrac{1}{2}\,\mathrm{Re}\,\{S(\omega + \Omega)(c' - ic'' + d'' + id') \times \exp[-i(\omega + \Omega)t] + S(\omega - \Omega) \times (c' + ic'' - d'' + id') \exp[-i(\omega - \Omega)t]\} \tag{56}$$

A much more concise notation consists in writing

$$A(t) = \mathrm{Re}\,\mathrm{Re}_j\{A_{bc} \exp(-i\omega t + j\Omega t)\}$$

$$A_{bc} \equiv c' + jc'' + i(d' + jd'') \tag{57}$$

where Re_j has the same meaning as 'Re' but refers to 'j' rather than 'i'.

With that notation, eqn. 56 can be written as

$$B(t) = \mathrm{Re}\,\mathrm{Re}_j\{B_{bc} \exp(-i\omega t + j\Omega t)\} \tag{58a}$$

where

$$B_{bc} = S(\omega + ij\Omega)A_{bc} \tag{58b}$$

To prove that eqns. 56 and 58 are equivalent, it suffices to expand S in power series of Ω, assuming that S is analytic in a sufficiently large domain about ω, and consider separately even and odd powers of Ω. The calculations are lengthy but straightforward.

14 Appendix B. Standard rate equations

The standard rate equations (SRE) keep track of the number of electrons in the semiconductor and of the number of photons in the cavity as a function of time. We consider here the formulation given by Agrawal [2], modified or clarified as follows.

(i) Nonlinear gain is not considered, and thus we set: $\partial G/\partial P = 0$.

(ii) Terms depending on the rate R_{sp} of spontaneous emission in the oscillating mode are very small and are omitted. They do not appear in a consistently linearised theory.

(iii) It is considered implicit that the driving circuit generates shot noise.

(iv) The notation is converted to the one used in this paper as follows:

$$G \to 2\pi f_0;\ G_N \to 2\pi f_0 G_{1N}/G_1;\ P \to P/2\pi f_0$$

$$\gamma_e N \to S;\ \gamma_e + N d\gamma_e/dN \to S_N$$

$$R_{sp} \to 2\pi f_0 n_s \equiv n_s/\tau_p$$

$$2D \to S \text{ (double-sided spectral densities)}$$

$$\delta\phi \to -\delta\phi$$

$$\delta P \to \delta P/2\pi f_0 \tag{59}$$

The latter two transformations need comments. In SRE δP refers to intensity noise only. The prescription therefore is made that one must add to the detector-current spectral density resulting from the intensity noise δP an independent term (equal to P in our notation). This term is interpreted physically as being the result of the detector-current shot-noise, or, alternative, as being caused by the photon noise. (The SRE prescription, incidentally, precludes that the detector current fluctuation could be less than shot noise). Our δP has a different meaning: it gives directly the total detector-current fluctuation, and may be less than shot noise, as is also predicted by quantum optics for light beams with 'subpoissonian photon statistics'. The SRE prescription just discussed is incomplete as far as the phase is concerned. In order to reach exact agreement with quantum optics one must also add a term $1/4P$ (in our notation) to the phase spectral density predicted by the SRE.

It would not make sense to compare our rate equations with the SRE since the quantities δP, $\delta\phi$ do not have the same physical meaning in the two formulations. What we can do is compare the predictions of the two theories for measurable quantities, such as the detector-current spectral density.

Eqns. (6.5.4–10) of Reference 2 become in our notation

$$j(f/f_0)\delta P = gP\delta N/N + F_P$$

$$j2\pi f\,\delta N = -(s\zeta + g)P\delta N/N - \delta P + F_N$$

$$2j(f/f_0)P\,\delta\phi = -\alpha g P\,\delta N/N + F_\phi$$

$$S_{FP} = 2n_s P \quad S_{FN} = 2n_s P + 2S$$

$$S_{FP\,FN} = -2n_s P \quad S_{F\phi} = 2n_s P \tag{60}$$

and the other cross-correlations vanish.

If we eliminate δP from the two first equations in eqn. 60 we find

$$PD\delta N/N = F_N - \frac{f_0}{jf} F_P \tag{61}$$

where D was defined in eqn. 21. The spectral density of the process on the right-hand-side of eqn. 61 is the same as the one of our noise term $s_n + e_n + p_n$ when $\xi = 1$, and thus eqn. 61 coincides with our result for that case.

Consider next the optical power fluctuations. We have from the previous equations

$$\delta P/P = (\zeta r/j2\pi f ND)[(s\zeta + g + jxr\zeta)F_P + gF_N]$$

$$S_{\delta P} = P + 2P[\zeta g^2 + n_s \zeta^2(s^2 + r^2x^2)]/x^2|D|^2 \tag{62}$$

The first term, P, does not follow from the preceeding expression of δP. It was added according to the SRE prescription. (In eqn. (6.5.19) of Reference 2 the cross-correlation between F_N and F_P was inadvertently omitted, and a different result is given).

It is a simple matter to obtain the spectral density of the phase fluctuation since F_ϕ is not correlated with δN. The result is the same as in the main text (again with $\xi = 1$), except for the term $1/4P$ which must be added ad hoc to the SRE result, as we discussed above. We have verified that all the expressions given in the main text, including the cross-correlations, are the same as those that can be derived from the above SRE if $\xi = 1$, and the amplitude and phase prescriptions discussed above are made (addition of P and $1/4P$ to the power and phase spectral densities, respectively).