Whispering-gallery modes of dielectric resonators

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Indexing terms: *Dielectrics, Microwave components*

Abstract: The whispering-gallery modes of cylindrical dielectric resonators of high permittivity ($\epsilon_r \sim 38$) are investigated. Axial-mode confinement is ensured by a slight increase in the rod diameter. The resonances for various azimuthal mode numbers and the two states of polarisation are identified experimentally. There is a good agreement between measured and calculated frequencies. The measured quality factor, of the order of 7000 at 6 GHz, is, for some resonances, limited only by the intrinsic loss of the dielectric material. The whispering-gallery mode of resonance enables designers to use comparatively larger-size dielectric resonators at millimetre-wavelengths with good frequency selection and low loss.

1 Introduction

High-permittivity resonators have been studied in detail in recent years as useful microwave filters [1, 2]. This type of resonator offers the advantage of a small size and a good field concentration that facilitates coupling to active devices. The quality factor of a dielectric resonator is limited, on the one hand, by the intrinsic loss of the material and, on the other hand, by radiation losses.

For the high permittivity materials that are currently available, the intrinsic Q-factor is usually of the order of 7000. But this is often reduced to 3000 or less by radiation mechanisms.

The mode of resonance that is under investigation – the whispering-gallery mode – was first discovered in the field of acoustics by Lord Rayleigh [3]. This mode can be described as comprising waves running against the concave side of curved boundaries. In the present situation where we consider circular dielectric cylinders, the waves move essentially in the plane of the circular cross-section, and are confined by the dielectric-air discontinuity. From a ray-optics point of view, one can say that the rays are totally reflected at the dielectric-air boundary. More precisely, the wave is essentially confined between the cylindrical boundary and an inner caustic whose radius can be calculated from a modified ray-optics technique. The ray theory, however, proves insufficiently accurate for the present device. Therefore, the theory presented in this paper will be based on the well known exact formulas for the modes of propagation along dielectric cylinders in free space [4]. Let us recall that these modes are defined by an azimuthal mode number that we shall denote $\mu = 1, 2, \ldots$, and a radial mode number denoted $\alpha = 0, 1, 2 \ldots$. The theory provides the propagation constant h along the cylinder axis at some real angular frequency ω.

As a first approximation, the resonant frequencies of the dielectric resonator presently considered are obtained by setting $h = 0$ and looking for ω, μ being a large integer. (Strictly speaking the ω solution is complex but we shall neglect the imaginary part.) Note, however, that we are interested in waves that remain confined to a small region in the axial direction. This axial confinement is achieved by slightly increasing the diameter of the dielectric rod in some central region. In that configuration, the field of the resonant mode decays exponentially in the axial direction on both sides of the enlarged region. A simple theory that accounts for the axial confinement consists of matching the field and its first derivative at the diameter discontinuities as was carried out in References 5 and 6. As we shall see, these theoretical results are in good agreement with the experimental results, for both states of polarisation.

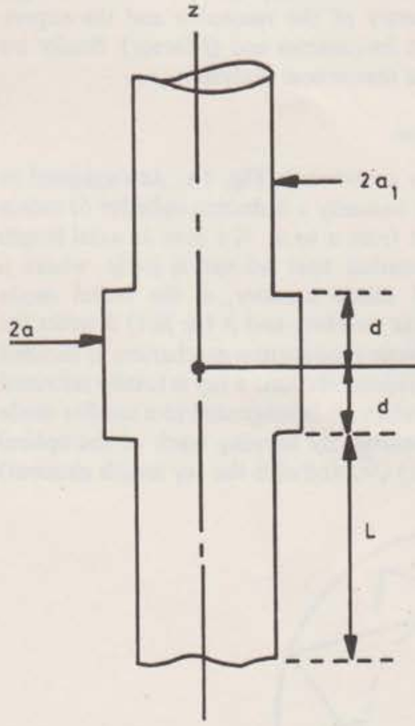

Fig. 1A *Schematic view of dielectric resonator*

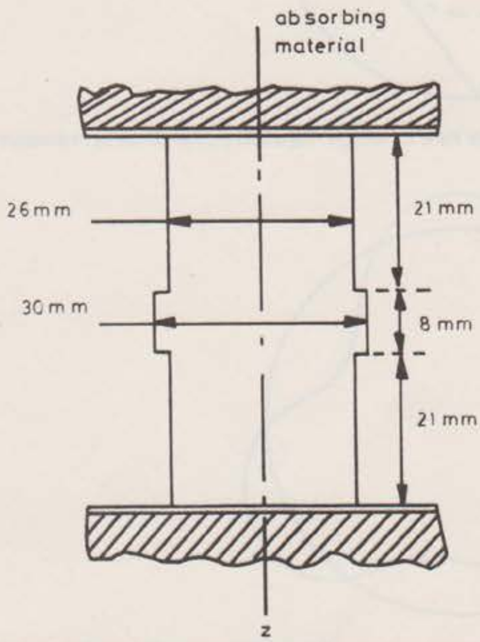

Fig. 1B *Dimensions of resonator under test*

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The whispering-gallery mode of resonance of high-permittivity dielectric resonators appears to be mostly useful in the millimeter-wave region, where more conventional dielectric resonators are impractically small. In spite of their large size, whispering-gallery-type resonators afford good suppression of spurious modes (e.g. modes with radial numbers $\alpha > 0$) because these unwanted modes leak out axially and are absorbed without perturbing much the desired modes. On the other hand, radiation losses are found to be almost negligible in the wavelength range considered, the quality (Q-) factor being almost as high as that of the intrinsic material, of the order of 7000 at 6 GHz.

These WG modes can be excited, for example, by synchronous external travelling waves supported by slow wave structures. In particular, we have experimented with a meander line photoetched on copper-plated Teflon.

In this paper, we shall first clarify the mode terminology; then describe the geometry of the resonator and the experimental results (resonant frequencies and Q-factor); finally we present the details of the theoretical analysis.

2 Mode denomination

The resonator geometry is shown in Fig. 1A. As explained in the introduction, this is basically a dielectric cylinder of radius a. The radius is reduced from a to $a_1 < a$ over an axial length $2d$. The mode denomination that we use is $\mu\alpha lp$, where μ denotes the azimuthal mode number, α the radial mode number, l the axial mode number, and p (or pol) denotes the polarisation state. The basic propagation mechanism is recalled in Fig. 2 in a ray-optics representation: a ray is totally reflected at the dielectric-air boundary. It is tangential to a smaller circle of radius b called the caustic. By keeping track of the optical phase ($k\,dl$, if $k = (\omega/c)\sqrt{\epsilon_r}$ and dl is the ray length element) along either the caustic or a ray, an approximate expression for the resonance condition can be obtained. In particular, we have $2\pi bk = \mu$, where μ denotes, as indicated above, the azimuthal mode number.

From a wave-optics point of view, μ is the number of wavelengths around the cylinder, as shown schematically in Fig. 2B. This number is, of course, independent of radius. The phase velocity of the mode in the azimuthal direction increases in proportion with radius, as is the case for rotating wheels. Note further that μ is a positive or negative integer, depending on the sense of rotation of the ray. However, because the dielectric medium is isotropic, the resonant frequencies are the same for $+\mu$ and $-\mu$. In other words, these two modes are degenerate. (The degeneracy would be lifted ($\omega_\mu \neq \omega_{-\mu}$), e.g. if the cylinder were submitted to an axial twist, through the photoelastic effect, i.e. the effect of strain on the permittivity tensor. Such strains preserve the invariance of the system under rotation.) It is therefore permissible to superimpose these two modes to generate $\cos\mu\phi$ or $\sin\mu\phi$ dependence of the field on the azimuth. As a matter of fact, most of our experiments were made with electric or magnetic probes that excite such field variations. However, we have also experimented on travelling-wave excitations capable of exciting independently either $\exp(i\mu\phi)$ or $\exp(-i\mu\phi)$ field variations. We are mostly interested in high μ values.

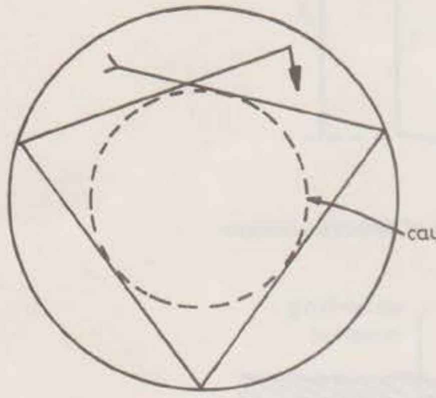

Fig. 2A *Ray-optics view of whispering-gallery-type mode of resonance*

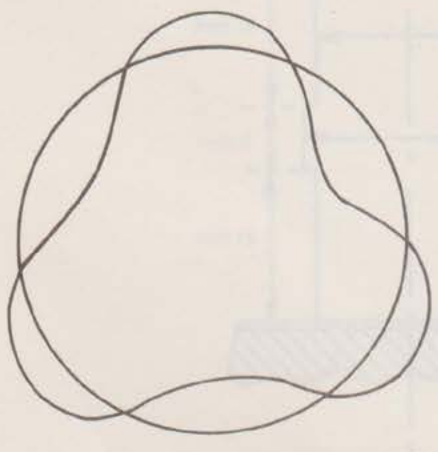

Fig. 2B *Schematic representation for field at azimuthal mode number $\mu = 3$*

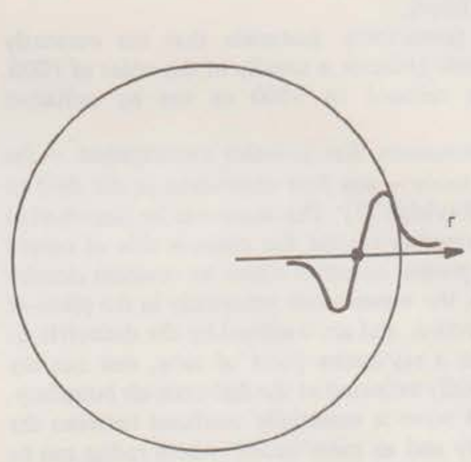

Fig. 2C *Radial field variation for spurious mode $\alpha = 1$*

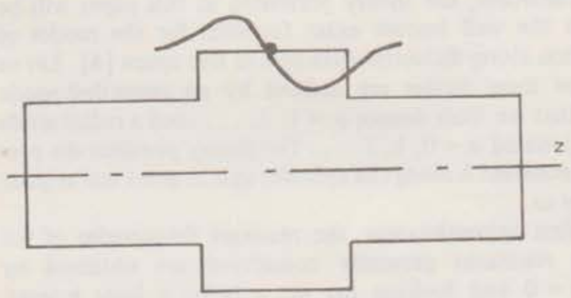

Fig. 2D *Axial field variation for spurious mode $l = 1$*

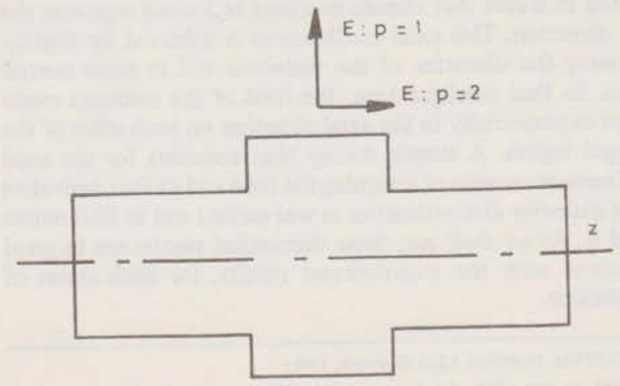

Fig. 2E *Definition of polarisation states*

The integer $\alpha = 0, 1, 2, \ldots$ denotes the radial mode number, i.e. the number of field nodes in the radial direction. In a very simplified model we may assume that the field is described by Bessel functions $J_\mu(kr)$ that vanish at the boundary $r = a$. The $\alpha = 0$ mode is such that the $J_\mu(kr)$ function has no zero from $r = 0$ to $r = a$. The $\alpha = 1$ mode corresponds to one zero, and so on, as shown in Fig. 2. We are mostly interested in $\alpha = 0$ modes. The other α values are viewed as being spurious.

The integer $l = 0, 1, 2, \ldots$ denotes the number of zeros (nodes) of the field in the axial direction. Thus, the symmetrical fundamental solution corresponds to $l = 0$. The first antisymmetrical mode corresponds to $l = 1$, and so on. We are mostly interested in the $l = 0$ modes, the other values being viewed as corresponding to spurious modes.

Finally, there are two possible states of polarisation of the electromagnetic field that we shall denote $p = 1$ and $p = 2$. For a wave motion strictly in the plane of the cross-section, the Maxwell equations show that either $E_z = 0$ ($p = 1$) or $H_z = 0$ ($p = 2$). In the former case, E is radial, and the mode can be excited by a small dipole antenna directed along the radial direction. In the latter case, E is axial, and the mode is most conveniently excited by a loop whose axis is oriented along the azimuthal direction.

The detailed electromagnetic theory has been relegated to Appendix 7.

3 Experiments

The geometry of the dielectric resonator is shown in Fig. 1B and the experimental set-up is shown in the photograph in Fig. 3. The dielectric resonator has a total height of 50 mm. Absorbing plates are located at both ends to absorb spurious modes that leak out from the central region.

The resonator is excited by a scanned microwave source and a short electric dipole. Similarly, the output is a short dipole and a broadband detector. For each resonance, one verifies that the field is concentrated near the enlarged section, and decays away from it.

The azimuthal mode number μ is determined unambiguously by rotating the detector probe around the cylinder axis. Finally the state of polarisation ($p = 1$: E along r; $p = 2$: E along z) is also unambiguously determined by using either an electric dipole or a small loop. Except for some unidentified signals at strong couplings, only the desired whispering-gallery-type resonances are observed, for all μ values from 1 to 9 and beyond. The Q-factor of a resonance is measured, as usual, from the 3 dB width at low coupling. The experimental results are summarised in Table 1. For the two polarisation states, we have shown in the Table the calculated resonant frequency, in gigahertz, assuming $\epsilon_r = 38$, the measured resonant frequency, and the measured Q-factor.

Table 1: Comparison between calculated and measured resonant frequencies for various μ values

μ	3	4	5	6	7	8	9	
F (GHz) theory	3.62	4.25	4.87	5.47	6.05	6.63	7.21	$E_z \simeq 0$ ($p = 1$)
F (GHz) exp.	3.71	4.37	5.02	5.63	6.25	6.82	7.44	
Q	1000	4120	3920	5410	4140	4110	4300	
F (GHz) theory	2.85	3.54	4.19	4.79	5.40	6.00	6.59	$H_z \simeq 0$ ($p = 2$)
F (GHz) exp.	3.06	3.77	4.45	5.10	5.74	6.35	6.96	
Q	1600	5990	7070	6200	6100	4670	5234	

Measured Q-factors ($\epsilon_r = 38$, $2a = 30$ mm, $2a_1 = 26$ mm, $2d = 8$ mm)

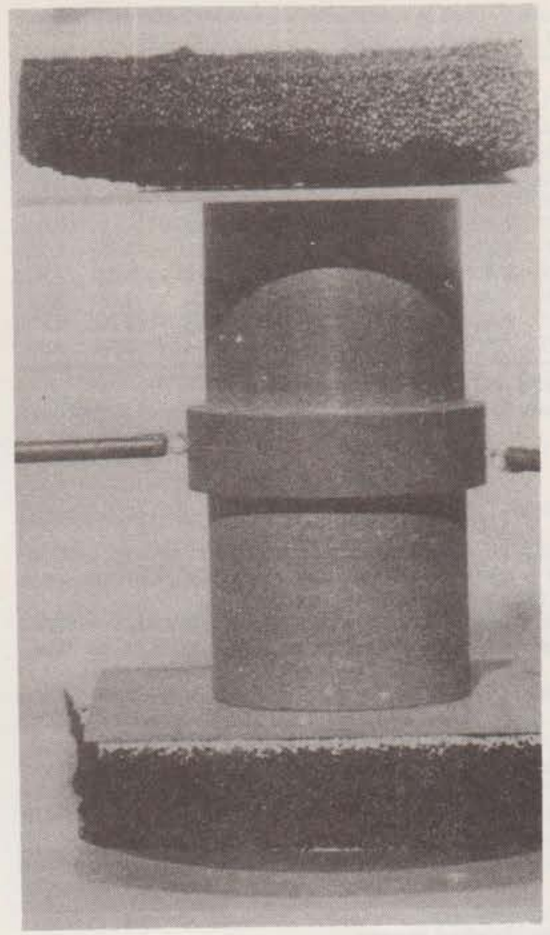

Fig. 3 *Photograph of experimental set-up*

The agreement between the approximate calculation, given in Appendix 7, and the measured values is fairly good. Part of the difference may be due to some uncertainty in the value of ϵ_r. As far as the Q-factor is concerned, the intrinsic Q_i of the material is said by the manufacturer to obey a law $Q_i f \simeq 40\,000$ in the relevant range of frequencies. Thus, at 4.5 GHz, $Q_i \simeq 8800$. The measured value $Q = 7070$ is close to it. This indicates that the radiation losses are small. At higher frequencies, the Q-factors drop, presumably because of increasing material losses. At lower frequencies, it also drops, presumably because of increased radiation losses.

4 Conclusion

The whispering-gallery-type mode of resonance in high-permittivity dielectric resonators has been observed unambiguously. The resonant frequencies are in good agreement with values calculated from an approximate theory. The Q-factors are high, about 7000, and probably limited by material loss. Because of the high modal purity, this type of resonator should be useful as a filter element in the millimetre-wave range of frequencies. A number of such resonators can be coupled together either axially, with a number of enlarged sections along a single rod, or by placing them side by side, to get sharper band edges.

5 Acknowledgments

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Table 2: Calculated resonant frequencies for various μ values (columns) and various α, l and p mode numbers

μ \ α, l, pol	001	002	011	012	101	102	111	112
2	2.94	2.12			4.71	3.75		4.11
3	3.62	2.86			5.40	4.53		4.96
4	4.25	3.54			6.06	5.22		5.71
5	4.27	4.19			6.71	5.97		6.54
6	5.47	4.79			7.35	6.61	8.05	7.20
7	6.06	5.40		5.90	7.96	7.26	8.70	7.92
8	6.63	6.00	7.25	6.58	8.57	7.90	9.35	8.62
9	7.21	6.59	7.90	7.25	9.19	8.52	10.02	9.25
10	7.76	7.16	8.50	7.82	9.79	9.74	10.68	9.96

$\epsilon_r = 38$, $2a = 30$ mm, $2a_1 = 26$ mm, $2d = 8$ mm
Blanks correspond to nonresonating modes (the modes leak out axially)

6 References

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7 Appendix: Detailed theory

The object of this Section is to determine the resonant frequencies of the dielectric cylinder shown in Fig. 2, with its enlarged diameter. The dissipation loss of the high-permittivity dielectric cylinder will be neglected here ($\tan \delta \simeq 0$). All dielectric resonators exhibit radiation losses. When the resonator size is comparable with the wavelength in the material, these radiation losses significantly reduce the usefulness of the resonator as a filtering circuit element. But as the resonator size gets larger and larger, the radiation losses are drastically reduced for some modes. In considering the whispering-gallery-type modes than run along the dielectric cylinder boundary, and cylinder diameters large compared with wavelength, the radiation losses are indeed very small. In this Section, we shall neglect the radiation losses, and concentrate on the calculation of the real part of the resonant frequencies.

Let us consider first a uniform dielectric cylinder of radius a, and relative permittivity $\epsilon_r \gg 1$, immersed in free space. We consider propagating modes of the form

$$E(r,\phi,z,t) = E(r)\exp\{i(\mu\phi + hz - \omega t)\} \quad (1)$$

where the $E(r)$ function is concentrated near the cylinder boundary $r \simeq a$. The exact analytical relationship that exists between ω and h, for some azimuthal mode number μ, is well known. We have

$$(\eta_1 + \eta_2)(\epsilon_r\eta_1 + \eta_2) = \mu^2 h^2/(\omega/c)^2\,(u^{-2} + w^{-2})^2 \quad (2a)$$

where

$$\eta_1 = J'_\mu(u)/\{uJ_\mu(u)\};\ \eta_2 = K'_\mu(w)/\{wK_\mu(w)\}$$

and

$$(u/a)^2 = (\omega/c)^2\epsilon_r - h^2;\ (w/a)^2 = h^2 - (\omega/c)^2 \quad (2b)$$

As a first approximation, we assume that $h = 0$, i.e. we neglect the axial motion of the waves. The ω solutions of eqns. 2, rigorously speaking, are complex because w is imaginary. However, neglecting the radiation losses, we can take the real part of both sides of eqn. 2a and find approximate real values for ω, ignoring the small imaginary parts on both sides of eqn. 2.

Under the $h = 0$ approximation, the left-hand side of eqn.2 splits into two equations, one for each state of polarisations. A detailed analysis shows that the equation

$$\eta_1 + \eta_2 = 0 \quad (3)$$

corresponds to an electric field directed essentially in the radial direction. The other equation

$$\epsilon_r\eta_1 + \eta_2 = 0 \quad (4)$$

corresponds to an electric field directed essentially in the axial direction. In that way, one gets two sets of resonant frequencies F_0, one for each polarisation state.

To account for the central cylinder enlargment, we need to consider small, but nonzero, values of h. Using a numerical technique we make a plot of the frequency F as a function of h^2 for some permittivity ϵ_r and cylinder radius a. Such a curve is shown in Fig. 4, for $\epsilon_r = 38$ and $2a = 30$ mm. For small h, this curve is essentially a straight line that can be written in the form

$$F = F_0 + \theta h^2 \qquad (\text{radius } a) \quad (5)$$

where F_0 and θ are constants.

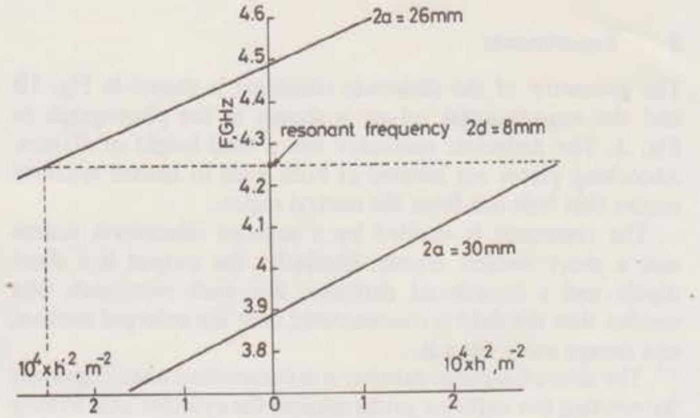

Fig. 4 *Variation of frequency of source exciting uniform dielectric rod*

$\epsilon_r = 38$, rod diameter = 30 mm (lower line) and rod diameter = 26 mm (upper line) as a function of the square of the propagation constant h. Negative values of h^2 (denoted here h'^2) correspond to evanescent waves. For the enlargment width $2d = 8$ mm, the resonant frequency is given by the horizontal line. The electric field is essentially radial: mode 4001.

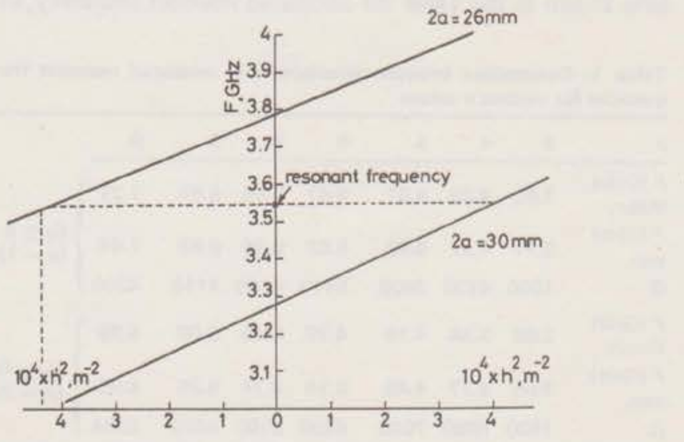

Fig. 5 *Similar to Fig. 4 with electric field essentially axial: mode 4002*

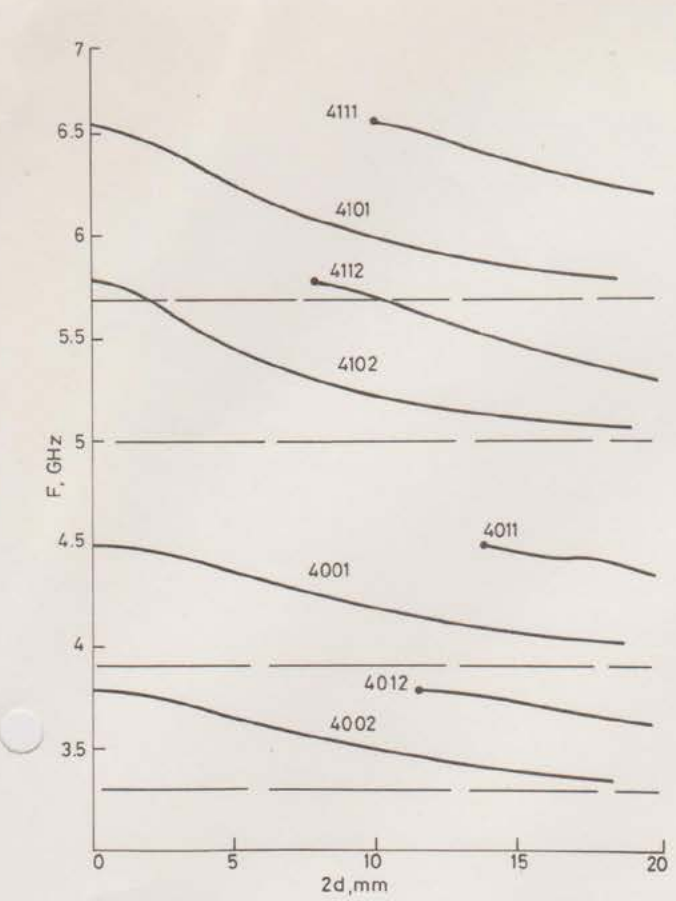

Fig. 6 *Calculated resonant frequencies for azimuthal mode number $\mu = 4$ and $\alpha = 0$ or 1, $l = 0$ or 1, and two polarisation states, as function of enlargement width 2d*

These curves enable us to select $2d$ in such a way that $\alpha \neq 0$, $1 \neq 0$ modes do not resonate, or are strongly attenuated.

For propagating waves $h^2 > 0$. However, we shall later be interested also in evanescent waves that correspond to $h^2 < 0$.

If the cylinder radius is $a_1 = a/\alpha$ with $\alpha \neq 1$ instead of a, we need not perform the calculations again. Using simple scaling laws, we find that the relationship

$$F = \alpha F_0 - (\theta/\alpha)h'^2 \qquad (\text{radius } a_1 = a/\alpha) \quad (6)$$

holds, where we denote the axial propagation constant h' instead of ih for later convenience.

Let us now consider the geometry in Fig. 1A. The cylinder has radius a in the central region over an axial length $2d$, and a smaller radius a_1 ($\alpha > 1$) outside that region. The approximate technique we have used consists of matching the field and its first derivative at the discontinuity. Taking the origin at the centre of the enlarged section, for symmetrical modes, the field has the form

$$\psi(z) = A\cos(hz) \qquad |z| < d \quad (7a)$$

$$\psi(z) = B\exp(-|h'|z) \qquad |z| > d \quad (7b)$$

The resonant frequency F is such that $h^2 > 0$, $h'^2 > 0$ (evanescent waves). By matching ψ and $d\psi/dz$ at $z = d$, and using the law in eqns. 5 and 6 found from the numerical solution of eqn. 2, we find the resonant frequencies F by solving the equation

$$\sqrt{\frac{F-F_0}{\theta}}\, d \tan\left(\sqrt{\frac{F-F_0}{\theta}}\, d\right) - \sqrt{\frac{\alpha}{\theta}(\alpha F_0 - F)}\, d = 0 \quad (8)$$

A similar relation can be obtained for antisymmetrical modes.

Plots of the $F(h^2)$ lines are shown in Figs. 4 and 5 for radial and axial electric fields, respectively. The resonant frequencies of various modes of low order are shown in Fig. 6 as a function of the enlargment width $2d$.