

2.2 Theory of Bar Lines

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I. Transverse Modes on Infinite Arrays of Bars

We will study in this section the electromagnetic waves which can be guided by an array of parallel cylindrical conductors directed along Ox .

A great number of practically interesting structures for crossed-field traveling wave tubes can be considered as resulting from the junction of many uniform sections constituted by such arrays. The computation of their dispersion characteristics and of their coupling impedance can be made in a somewhat general way, after the study of the uniform array.

Let us consider an array of infinite cylindrical conductors called "bars" directed along Ox . If they are immersed in a medium of constants ϵ_0 and μ_0 over all space, transverse electric and magnetic waves can propagate along the bars (1). For these modes of propagation to which we limit ourselves Maxwell's equations and the boundary conditions are satisfied by the fields \mathbf{E} and \mathbf{H} if

$$\begin{aligned} E_y &= -\frac{\partial \phi_0}{\partial y} e^{-jkx} & H_y &= \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\partial \phi_0}{\partial z} e^{-jkx} \\ E_z &= -\frac{\partial \phi_0}{\partial z} e^{-jkx} & H_z &= -\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\partial \phi_0}{\partial y} e^{-jkx} \end{aligned} \quad (1)$$

$$E_x = 0 \quad H_z = 0$$

$$\frac{\partial^2 \phi_0}{\partial y^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad (2)$$

k is the constant of free propagation in the medium, ϕ_0 will be called the potential (it must be constant for given x and l on the surface of any bar), and V will be the potential of the bar for these values of x and l .

The integral of H on a contour enclosing one and only one bar and drawn in a plane of constant x will be called the current of that bar.

The general equations of multifilar lines can then be written (2) for any array of parallel conductors,

$$V_n = A_n \cos kx + B_n \sin kx$$

$$I_n = -j \sin kx \sum_m c \gamma_{mn} A_m + j \cos kx \sum_m c \gamma_{mn} B_m \quad (3)$$

where n and m index two given bars, and the sum extends to all the bars. The γ_{mn} coefficient is the capacity per unit length between bars m and n (unless otherwise mentioned we shall always deal with capacity per unit length in this paper), and A and B are arbitrary complex coefficients. The problem of the determination of the γ_{mn} 's is a two-dimensional electrostatic one and it will be studied later on, but it is sometimes easier to measure them, say, in an electrolytic tank.

The rms electrical energy stored per unit length on the n th bar is given by

$$U_n = \frac{1}{4} V_n \sum_m \gamma_{mn} V_m^* \quad (4)$$

If T_n is the corresponding magnetic energy it is possible to show from (3) that

$$U_n + T_n = \frac{A_n \sum_m \gamma_{mn} A_m^* + B_n \sum_m \gamma_{mn} B_m^*}{4} \quad (5)$$

is independent of x .

II. Periodic Array

We consider now an elementary cell consisting of N infinite bars and use it to generate an infinite array by translations

$$l\mathbf{p}_1 + m\mathbf{p}_2 \quad (6)$$

where \mathbf{p}_1 and \mathbf{p}_2 are two vectors of the plane Oyz , and l, m are integers taking all the values $0, \pm 1, \pm 2, \dots$ (Fig. 1).

We have here a double periodicity. In the usual cases (helix, interdigital line) we have one periodicity and $\mathbf{p}_2 = \infty$, $|\mathbf{p}_1| = p$. Since the interaction due to the double periodicity is only formal and can be useful for some new structures, we treat here the general case.

The N bars of the fundamental cell will be indexed by r or by $R = 1, 2, \dots, N$. The cells are indexed after the definition of the array by

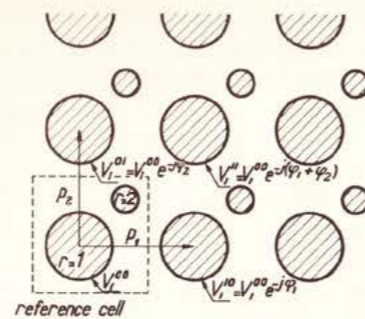


FIG. 1. Biperiodic bar type structure in the case $N=2$ bars in cell. \mathbf{p}_1 and \mathbf{p}_2 define the periodicity; the fundamental phase shifts for a given mode are φ_1 and φ_2 .

(l, m) . Let us call $\gamma_{r,R}^{(l,m)}$ the capacity between the bar r of the cell $(0, 0)$ and the bar R of the cell (l, m) ; we observe that

$$\gamma_{r,R}^{(l,m)} = \gamma_{R,r}^{(-l,-m)} \quad (7)$$

Then, we define N^2 characteristic admittances (3) by

$$Y_{rR} = c \sum_{l,m} \gamma_{r,R}^{(l,m)} \exp[-j(l\varphi_1 + m\varphi_2)] \quad (8)$$

where φ_1 and φ_2 are the fundamental phase shifts along \mathbf{p}_1 and \mathbf{p}_2 for a given mode; φ_1 and φ_2 can be introduced by Floquet's theorem; and c is the free propagation velocity.

From (7) we have

$$Y_{rR} = Y_{Rr}^* \quad (9)$$

(the asterisk indicates the conjugate value). Then, Y_{rr} can be written:

$$Y_{rr} = 4c \sum_{l,m>0} |\gamma_{r,r}^{(l,m)}| \sin^2 \left[\frac{l\varphi_1 + m\varphi_2}{2} \right] + c \sum_{R \neq r} \sum_{l,m} |\gamma_{r,R}^{(l,m)}| \quad (10)$$

In the case where the bars $R = \rho$ of each cell are connected together, they will be called the "ground" of the structure. For conciseness, we put

$$\sum_{l,m} |\gamma_{r,\rho}^{(l,m)}| = \gamma_r \quad (11)$$

γ_r will be called the capacity between the r th bar and the ground. The characteristic admittances Y are useful for one can use with them the multifilar line transformation formulas which give the voltages and the currents of a bar, in a cell, at $x = 0$, from their values at $x = d$, in the same cell.

$$||V_r(0)|| = \cos kd ||V_r(d)|| + j \sin kd ||Z_{rR}|| \cdot ||I_R(d)||$$

$$||I_r(0)|| = \cos kd ||I_r(d)|| + j \sin kd ||Y_{rR}|| \cdot ||V_R(d)|| \quad (12)$$

with

$$||Z_{rR}|| = ||Y_{rR}||^{-1}$$

We will apply these relations to a very simple case; we have one periodicity ($\mathbf{p}_2 = \infty$), and two bars only in a cell, one of them, namely $R = 1$, constituting the "ground," the other, namely $R = 2$, is referred to as the bar.

We are only interested in $Y = Y_{22}$ which can be written

$$Y = c\gamma_0 - 2c \sum_{l>0} \gamma^{(l)} + 2c \sum_{l>0} \gamma^{(l)} \cos l\varphi = c\gamma_0 + 4c \sum_{l>0} |\gamma^{(l)}| \sin^2 \left(\frac{l\varphi}{2} \right) \quad (13)$$

Relation (13) becomes†

$$V(0) = \cos kd V(d) + j \left(\frac{\sin kd}{Y} \right) I(d)$$

$$I(0) = \cos kd I(d) + j(\sin kd) Y V(d) \quad (14)$$

In this simple case γ_0 is used for γ_2 and $\gamma^{(1)}$ or γ' for $\gamma_{22}^{(1,0)}$

III. Calculation of the Characteristic Admittance and Space Harmonic Intensity

A. TAPE STRUCTURES

In this section we consider a structure which has great theoretical interest. It consists of infinitely thin bars in the same plane (see Babinet's

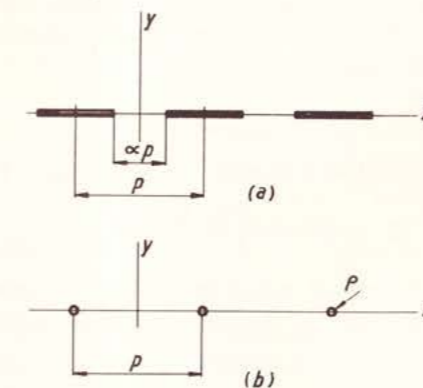


FIG. 2. (a) Tape structure of pitch p and gap ap ; the origin of the coordinates is in middle of a gap. (b) Thin wire structure of pitch p ; the radius of the wires is $\rho \ll p$; the origin of the coordinates is in the middle of two adjacent wires.

principle discussed in the previous Section 2.1 by this author, and Eq. (11) of that section). Its great interest is that it is possible, in this case, to ob-

† This results from the infinity of the capacities between the "ground" bars.

tain exact solutions for the fields, the characteristic admittance, and the coefficients $\gamma^{(l)}$.

Such a structure is shown in Fig. 2(a). It is obvious that in the plane of the structure ($y = 0$) the field is directed along Oz in the gaps and along Oy in the conductors. It can be shown (4) that it varies as:

$$E_z(0, z) = \frac{\exp[j(\pi - \varphi)z/p]}{(p/\pi)[\sin^2(\pi\alpha/2) - \sin^2(\pi z/p)]^{1/2}} \quad (15)$$

where αp is the gap width, the origin $z = 0$ being at the middle point between two adjacent bars. The field varies at the tape edges as $(z - \alpha p/2)^{-1/2}$. Now, for simplicity, we shall consider only the case where $\alpha = \frac{1}{2}$ (gap width equal to the tape width),

$$E_z(0, z) = \frac{\exp[j(\pi - \varphi)z/p]}{[\cos(2\pi z/p)]^{1/2}} \quad (16)$$

Let us suppose that the potential of the tape which is just at the left of the origin is $e^{j\varphi/2}$; the integral of E_z from $z = -\alpha p/2$ to $\alpha p/2$ must be equal to $2j \sin(\varphi/2)$. Then, we have

$$E_z = \frac{2j \sin \varphi/2}{p} \frac{\sqrt{2}}{P_{-\varphi/2\pi}(0)} \exp\left[\frac{j(\pi - \varphi)z}{p}\right] \left[\cos \frac{2\pi z}{p}\right]^{-1/2} \quad (17)$$

where $P_\nu(\cos \theta)$ is a Legendre polynomial generally of nonintegral order. It can be expressed with help of the Γ function:

$$P_\nu(0) = -\frac{\sin \nu\pi}{2\pi\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \quad (18)$$

Such a field can be analyzed in the space harmonics of magnitude

$$E_{zm} = \frac{2j \sin \varphi/2}{p} \frac{P_m(0)}{P_{-\varphi/2\pi}(0)} \quad 0 < \varphi < 2\pi \quad (19)$$

E_{zm} is not a continuous function of $\varphi + 2m\pi$ as shown in Fig. 3, curve (3).

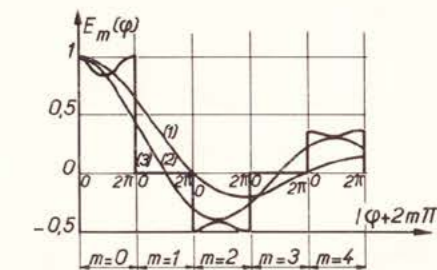


FIG. 3. Magnitude of the space harmonics in a tape structure (pitch unity, voltage unity between adjacent conductors). Curve (1), field assumed constant in the gap; curve (2), field assumed to be the same as for two semi-infinite planes; curve (3), exact solution.

This would imply the existence of a sheet of charge and current. The electric field of each space harmonic has no divergence (see the previous Section 2.1 on "General Properties of Periodic Structures" by this author). We can thus deduce E_{ym} from E_{zm} and calculate the current components $I_{sz,m}$ from E_{ym}

$$I_{sz,m} = -j \sqrt{\frac{\epsilon_0}{\mu_0}} \text{signum}(m) E_{zm} \quad (20)$$

Now, it is possible to sum over m and obtain the current density which obviously must vanish in the gap. Furthermore, it can be integrated over the tape; this gives half the tape current since the tape has two sides. This current is equal to the characteristic admittance

$$Y = \sqrt{\frac{\epsilon_0}{\mu_0}} \left| 4 \sin \frac{\varphi}{2} \right| \quad (21)$$

This simple result can be derived directly from Babinet's principle if we note that E_y has the same variation as E_z , but the more complicated computation indicated above can be useful to obtain an approximate value of Y when we have a ground parallel to the structure (5).

Using relation (13) one sees that $\gamma^{(1)}$ can be deduced from Y by a Fourier series expansion; then, we have

$$\gamma^{(1)} = \epsilon_0 \frac{2}{\pi} \frac{1}{l^2 - \frac{1}{4}} \quad (22)$$

$$\gamma^{(1)} = 7.5 \frac{pf}{m}; \quad \gamma^{(2)} = 1.5 \frac{pf}{m}; \quad \gamma^{(3)} = 0.65 \frac{pf}{m} \quad (23)$$

We then obtain Y in the form

$$Y = \sqrt{\frac{\epsilon_0}{\mu_0}} \left[\frac{8}{\pi} - \sum_{l=1}^{\infty} \frac{4/\pi}{l^2 - \frac{1}{4}} \cos l\varphi \right] \quad (24)$$

The characteristic impedance (21) is plotted in Fig. 4, curve (1); the

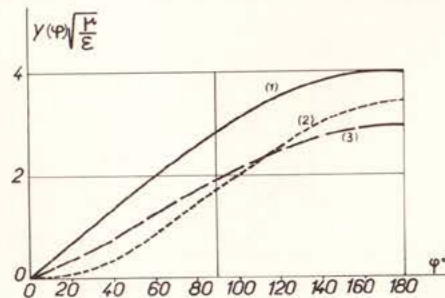


FIG. 4. Characteristic admittance of a tape structure. Curve (1), exact solution (deduced from the Babinet's principle); curve (2), expansion in Fourier series limited to the first term; curve (3), expansion in space harmonics with the assumption of a constant field in the gaps.

TAPE STRUCTURES	CHARACTERISTIC ADMITTANCE	
	SPACE HARMONIC MAGNITUDE	ASSUMPTION
	$E_m = \frac{\sin \left[\frac{1}{4} (\varphi + 2m\pi) \right]}{\frac{1}{4} (\varphi + 2m\pi)}$ curve 3(1)	$E_z(z) = 2$ $-\frac{1}{4} < z < \frac{1}{4}$
	$E_m = j_0 \left[\frac{1}{4} (\varphi + 2m\pi) \right]$ curve 3(2)	$E_z(z) = \frac{1}{\pi} \sqrt{\frac{1}{16} - z^2}$ $-\frac{1}{4} < z < \frac{1}{4}$
	$E_m = \frac{P_m(\varphi)}{P_g(\varphi)} \cdot \frac{2}{2\pi}$ curve 3(3)	$E_z(z) = \frac{\sqrt{2} e^{-j(\pi-\varphi)z}}{\cos 2\pi z} \cdot \frac{P_g(\varphi)}{2\pi}$ $-\frac{1}{4} < z < \frac{1}{4}$
	$\sqrt{\frac{\mu}{\epsilon}} = 2 \sin \frac{\varphi}{2} \sum_{n=1}^{\infty} (-1)^n \text{sign}(m) j_0^2 \left[\frac{1}{4} (\varphi + 2m\pi) \right]$ curve 4(1)	$\gamma' = \frac{8}{3\pi} \epsilon_0$ $\gamma \neq 1 = 0$
	$\sqrt{\frac{\mu}{\epsilon}} = 2 \sin \frac{\varphi}{2} \sum_{n=1}^{\infty} (-1)^n \text{sign}(m) j_0^2 \left[\frac{1}{4} (\varphi + 2m\pi) \right]$ curve 4(2)	

FIG. 5. Some theoretical results about tape structures, space harmonic magnitude, and characteristic admittance (pitch unity, voltage unity between adjacent conductors).

method of space harmonic expansion with the assumption of a constant field in the gaps gives inaccurate results as can be seen in Fig. 4, curve (3). This method is meaningless in the case where there is a ground plate very near to the structure. The expansion (24) in capacities limited to the first term gives the admittance plotted in Fig. 4, curve (2). A table of these main results is given in Fig. 5.

B. THIN WIRE STRUCTURES

We consider here an array of thin cylinders of radius ρ very small compared to the pitch (Fig. 2b). As a first approximation we can assume that the field created by the charges on a bar is the same as if the total charge was concentrated at its center. Then, we have to consider the field created by a set of line charges at $z = \pm(2n-1)p/2$. Each charge q creates a radial field $q/2\pi\epsilon_0 r$ if r is the distance between the point and the charge.

The summation over all the charges can be made easily in some cases, using the mathematical identity:

$$\sum_{n=1,3,5}^{\infty} \frac{\cos nx}{n^2 - \mu^2} = \frac{\pi \sin \mu(\pi/2 - x)}{4\mu \cos(\mu\pi/2)} \quad (25)$$

and its derivative with respect to x . For $y = 0$ we have only a z component,

$$E_z = \frac{q}{2p\epsilon_0 j} \frac{\exp[(jz/p)(\pi - \varphi)]}{\cos(z/p)\pi} \quad (26)$$

which could also be derived from the previous expression (15) for $\alpha = 1$. In this case the field becomes infinite near the charges as $\{z - (p/2)\}^{-1}$. All the space harmonics have the same magnitude at $y = 0$, where we have the spectrum of a series of sharp pulses. Furthermore, all the $\gamma^{(1)}$'s are equal (and very small).

By summation we can also obtain, for instance, the y component of the field at $z = 0$,

$$E_y = \frac{q}{2p\epsilon_0} \frac{\sinh(y/p)(\pi - \varphi)}{\cosh(y/p)\pi} \quad (27)$$

which is asymptotic for large y to

$$E_y \simeq \frac{q}{2p\epsilon_0} e^{-(\varphi/p)y} \quad (28)$$

Practically, the only important point to mention is the following: if we are seeking the higher coupling impedance, i.e., the minimum stored energy for a given field of the fundamental, the thin wire structures are not satisfactory because large amounts of energy are stored near the wire; but such a structure creates high order space harmonics, as well as its complementary, the thin gap structure.

C. RECTANGULAR SHAPE CROSS SECTION

Let h be the height of the bars and αp the gap width. When h is very great in comparison with αp we can neglect the fringing fields and we have

$$\frac{\gamma'}{\epsilon_0} = \frac{h}{\alpha p} \quad (29)$$

$$\frac{\gamma_0}{\epsilon_0} = \frac{(1 - \alpha)p}{b} \quad (30)$$

where b is the distance between the ground and the structure ($b \ll p$). A better approximation can be made by taking into account the γ 's obtained in the infinitely thin structure analysis and writing (in the absence of ground and for $\alpha = \frac{1}{2}$)

$$\frac{\gamma'}{\epsilon_0} = \frac{2h}{p} + \frac{8}{3\pi} \quad (31)$$

and in the case of a ground near the structure

$$\frac{\gamma'}{\epsilon_0} = \frac{2h}{p} + \frac{4}{3\pi} \quad (32)$$

D. CIRCULAR CROSS SECTION

We can consider the capacity between two cylinders of diameter ϕ in free space; then, if $(b + \phi/2)$ is the distance between the cylinder axis and the ground:

$$\frac{\gamma'}{\epsilon_0} = \frac{\pi}{\cosh^{-1}(p/\phi)} \quad (33)$$

$$\frac{\gamma_0}{\epsilon_0} = \frac{2\pi}{\cosh^{-1}\left(1 + \frac{2b}{\phi}\right)} \quad (34)$$

These results are valid for ϕ close to p and small b .

E. SPACE HARMONICS FOR CONSTANT ELECTRIC FIELD ACROSS THE GAPS

The simplest computation of space harmonics assumes that the field is constant and equal to $V/\alpha p$ in the gap αp . This leads to

$$E_m = \frac{V \sin[(\alpha p/2)(\varphi + 2m\pi)]}{p(\alpha p/2)(\varphi + 2m\pi)} \quad (35)$$

It is valid for lower order space harmonics as can be seen in Fig. 3.

F. SPACE HARMONICS IN THE CASE OF FIELDS VARYING,
e.g., TWO SEMI-INFINITE PLANES

The Schwartz transform permits one to compute exactly the field between two semi-infinite planes. This field gives a better approximation than the previous one for a tape structure as shown in Fig. 3, curve (2).

List of Symbols

f	frequency
c	velocity of light
ω	angular frequency
$k (= \omega/c)$	propagation constant
$\lambda (= c/f)$	free space wavelength
ϵ_0	vacuum permittivity
μ_0	vacuum permeability
x, y, z	coordinates
t	time
$p, pi(i = 1, 2)$	itches of periodicity of the line
$\varphi, \varphi i(i = 1, 2)$	fundamental phase shift
E, \mathbf{E}	electric field
H, \mathbf{H}	magnetic field-axial vector
T	stored magnetic energy
U	stored electric energy
ϕ_0	potential
V	potential of a bar
I	current in a bar
d	length in direction of bar
$\gamma_{m,n}$	capacitance/unit length between bar n and bar m
γ_r, γ_0	capacitance/unit length between bar and ground
A, B	arbitrary voltages
l, m, n	integers
R, r, ρ	integers defining positions of a bar in a cell
$ Y_{r,R} $	generalized characteristic admittance
$ Z_{r,R} = Y_{r,R} ^{-1}$	generalized characteristic impedance
α	ratio of gap width to pitch
$P_r (\cos \theta)$	Legendre polynomial
$\Gamma(1+x)$	gamma function
signum (x)	+1 for positive x -1 for negative x
q	electric charge/unit length
ϕ	diameter of a finger
h	height of a finger

b	distance between a finger and the ground
N	number of bars in a cell
I_{ss}	surface current density
r	radial distance

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