Representation of Gaussian beams by complex rays

Jacques Arnaud

This paper shows that a fundamental Gaussian beam propagating in a lenslike medium with cylindrical symmetry can be generated by the rotation about its axis of a skew ray which obeys the laws of geometrical optics. A complex representation: $X(z) = \xi(z) + j\eta(z)$, where $\xi(z)$ and $\eta(z)$ are the projections of the skew ray on two perpendicular meridional planes, is discussed. It is found that the beam radius is equal to the modulus of $X(z)$ and the on-axis phase to the phase of $X(z)$. Using this representation, we derive a general expression for the on-axis phase shift $\Delta\Phi$ experienced by a beam with an input complex beam parameter q through an optical system whose ray matrix is $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$: $\Delta\Phi$ = phase of $(A + B/q)$. When the beam is matched to the optical system (output $q = q$), $\Delta\Phi$ can be written $\cos^{-1}(A + D)/2$. This representation also provides a useful beam tracing method which is demonstrated and a simple interpretation for the known representation of Gaussian modes by ray packets.

I. Preface

This paper was first issued as an internal Bell Labs. memorandum, 10 Oct. 1968, and widely distributed within the Bell system. My basic ideas of representing a Gaussian beam by a complex ray or by a complex coordinate shift were published at that time in *Applied Optics* (Refs. 12 and 13), but the paper itself was not submitted for publication. The motivation for publishing now this 16-year old paper stems from a renewed interest in our beam tracing procedure (see, e.g., Ref. 21). Many details, particularly concerning the wave front, have never been published to my knowledge and seem to remain of practical interest. Note that the word "phase $\Delta\Phi$" used in the paper in fact stands for phase difference. The trivial geometrical phase shift has been omitted. In Secs. III and IV the simple astigmatic Gaussian beam case is treated explicitly, while in Secs. V–VII more usual circular Gaussian beams are considered. More recent works, discussing general astigmatism and more general complex-coordinate shifts, are briefly reviewed in the Appendix. The paper is presented here exactly in its original form except for the addition of the Appendix, this preface, and the figure captions.

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II. Introduction

A Gaussian beam can be defined by a complex beam parameter $q = q_0 + z$ where q_0 is a complex quantity and z is the axial coordinate.[1,2] The real part of q_0 gives the beam waist position and its imaginary part gives the beam waist radius. The beam radius, the phase front radius, and the phase of the on-axis field at an arbitrary plane are also simply related to q. Kogelnik[1] gave the general law for transformation of q through an arbitrary optical system defined by its ray matrix: q is transformed by the same law as the wave surface curvature radius $[X(z)]/(dX/dz)$ of a homocentric ray pencil. No distinct meaning was attached, however, to $X(z)$ or dX/dz separately for the representation of Gaussian beams. It is the purpose of this paper to show that a Gaussian beam can be represented by a complex ray $X(z)$ which formally obeys the laws of geometrical optics. This representation provides a general expression for the on-axis phase shift that the simple consideration of the complex beam parameter q fails to give directly. In free space, the on-axis phase shift is given by the phase of q. The on-axis phase shift experienced by a beam as it propagates through an optical system can consequently be obtained by steps, from lens to lens. This conventional procedure was used by Kogelnik[3] to calculate the resonant frequency of a linear cavity incorporating arbitrary optical elements. We will show that a more general result can be obtained directly from the proposed representation.

Bykov and Vainshtein[4] and Kahn[5] have shown that the profile of the fundamental mode of a cavity is the envelope of a properly launched ray which bounces back and forth on the end mirrors. Steier[6] generalized this result to the case of propagating modes and noted that the bisectrix of two rays at a given point intersects the optical axis at the center of curvature of the wave front. We will show that these properties readily result from the following observation: A Gaussian beam propagating in a lenslike medium with cylindrical symmetry can be generated by the rotation about its axis of a skew ray which obeys the laws of geometrical optics. The complex ray $X(z)$ mentioned above is a complex representation of this skew ray.

The representation of a Gaussian beam by two rays, the real and imaginary parts of $X(z)$, provides a method of tracing its transformation through an optical system using ordinary ray tracing. This method is simpler, in some cases, than the method based on the Smith chart[7] or on the lateral foci,[8,9] and it gives directly the beam profile.

The following discussion is restricted to fundamental Gaussian beams with simple astigmatism[10] and to orthogonal[11] astigmatic optical systems.

III. General Expression for the Field of a Fundamental Gaussian Beam

The conventional expression for the field of a fundamental Gaussian beam propagating in free space[2] can be written

$$\Psi(x,y,z) = \frac{1}{\sqrt{q_x(z)q_y(z)}} \exp\left\{-j\frac{k}{2}\left[\frac{x^2}{q_x(z)} + \frac{y^2}{q_y(z)}\right]\right\}, \quad (1)$$

where xyz is a rectangular coordinate system, z being the beam axis, and $k = 2\pi/\lambda$ is the free space propagation constant. A factor $\exp(-jkz)$ has been omitted in Eq. (1) for simplicity. The complex beam parameter in the xz plane $q_x(z)$ is a linear function of z:

$$q_x(z) = z - z_{x0} + j\frac{kw_{x0}^2}{2}, \quad (2)$$

where z_{x0} and w_{x0} are, respectively, the beam waist position and halfwidth. A similar expression holds for $q_y(z)$. Note that, when $q_x(z)$ and $q_y(z)$ are real ($w_{x0} = w_{y0} = 0$), Eq. (1) gives the field of an astigmatic ray pencil in free space, the beam waists reducing to focal lines.

In the more general case where the ray pencil is transmitted through an orthogonal[11] lenslike medium, the field can be written

$$\Psi(x,y,z) = \frac{1}{\sqrt{n(z)X(z)Y(z)}} \exp\left\{-j\frac{k}{2}n(z)\left[\frac{x^2}{q_x(z)} + \frac{y^2}{q_y(z)}\right]\right\}, \quad (3)$$

where $n(z)$ is the on-axis refractive index and $X(z)$ and $Y(z)$ are the equations of two rays limiting the ray pencil in the xz and yz planes, respectively. The term in front of the exponential in Eq. (3) simply results from the conservation of power since the pencil cross-section area is proportional to $X(z)Y(z)$. For simplicity, a phase factor $\exp[-j\int_0^z kn(z)dz]$ has been omitted. In Eq. (3), $q_x(z)$ is defined by

$$q_x(z) = \frac{X(z)}{dX/dz} \quad (4)$$

and $q_y(z)$ by a similar expression. In free space, $n(z)$ is equal to unity and $(dX/dz),(dY/dz)$ are constant since the rays are straight lines. Equation (3) then reduces to Eq. (1) within a proportionality factor.

We will show that Eq. (3) gives the field of a Gaussian beam propagating through an arbitrary orthogonal medium if $X(z)$ and $Y(z)$ are considered complex quantities. It turns out that the on-axis phase can be obtained from Eq. (3) without further specifying $X(z)$ and $Y(z)$.

IV. On-Axis Phase Shift Experienced by a Beam Through an Optical System

The phase of $\Psi(0,0,z)$ given by Eq. (3) is

$$\Phi(z) = -\tfrac{1}{2}[\text{phase of } X(z) + \text{phase of } Y(z)]. \quad (5)$$

Let us use this expression to calculate the on-axis phase shift experienced by a beam between the input and output planes of a lossless optical system characterized by ray matrices

$$M_x = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \text{ and } M_y = \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix}$$

in the xz and yz planes, respectively. If we denote the derivatives with respect to z by an upper dot and the output quantities by a prime, from Eq. (3) we get

$$\begin{aligned} \Delta\Phi \equiv \Phi' - \Phi &= -\tfrac{1}{2}\left[\text{phase of } \frac{X'}{X} + \text{phase of } \frac{Y'}{Y}\right] \\ &= -\tfrac{1}{2}\left[\text{phase of } \frac{A_x X + B_x \dot{X}}{X} + \text{phase of } \frac{A_y Y + B_y \dot{Y}}{Y}\right] \\ &= -\tfrac{1}{2}\left[\text{phase of }\left(A_x + \frac{B_x}{q_x}\right) + \text{phase of }\left(A_y + \frac{B_y}{q_y}\right)\right], \end{aligned} \quad (6)$$

where q_x and q_y are the input complex beam parameters.

In free space, with $A_x = A_y = 1$ and $B_x = B_y = z$, from Eqs. (6) and (2) we get

$$\Delta\Phi = -\tfrac{1}{2}\left[\text{phase of } \frac{q_x'}{q_x} + \text{phase of } \frac{q_y'}{q_y}\right] \quad (7)$$

which could have been derived directly from the form of Ψ in Eq. (1). In the simple case where the beam has a cylindrical symmetry and a beam waist radius w_0 at

$$z = 0 \left(q_x = q_y = \frac{j\pi w_0^2}{\lambda}\right),$$

the phase shift is given, from Eqs. (7) and (2), by the well-known expression[2]

$$\Delta\Phi = \tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right). \quad (8)$$

For application to optical cavities, we are mostly interested in the phase shift experienced by a matched beam ($q_x' = q_x$ and $q_y' = q_y$) through an optical system such that $n' = n$. q_x is given in that case by the self-consistency equation[1]

$$q_x = \frac{A_x q_x + B_x}{C_x q_x + D_x} \quad (9)$$

and q_y by a similar expression. Introducing the solu-

tion of Eq. (9) which has a positive imaginary part into the last Eq. (6), we get

$$\Delta\Phi = \tfrac{1}{2}\left[\cos^{-1}\frac{A_x + D_x}{2} + \cos^{-1}\frac{A_y + D_y}{2}\right] \quad (10)$$

if we remember that[11] $A_{x,y}D_{x,y} - B_{x,y}C_{x,y} = (n/n') = 1$. For a linear cavity with cylindrical symmetry, Eq. (10) reduces to the expression obtained by Kogelnik[3] using the method outlined in Sec. II:

$$\frac{\Delta\Phi}{2} = \cos^{-1}\sqrt{ad}, \quad (11)$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the one-way ray matrix of the cavity.[12,13]

Equation (10) shows that $\exp(\pm j\Delta\Phi)$ are the ray matrix eigenvalues.

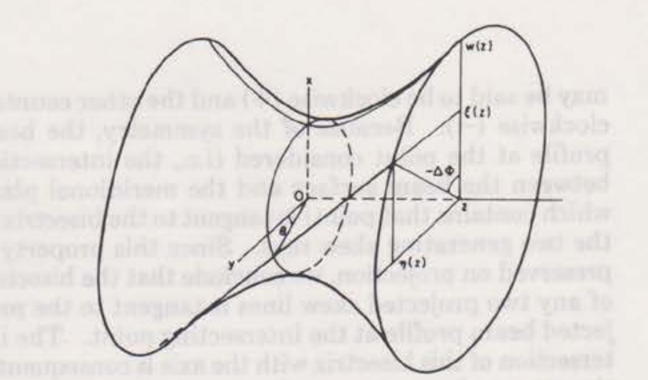

Fig. 1. Gaussian beam represented by a skew paraxial ray rotating about the z axis. At any z value, the beam radius is the distance between the ray and the z axis. Furthermore the angular position of the ray gives the difference between the phase of the optical field and the geometrical phase shift.

V. Beamwidths

Let us consider first a Gaussian beam with cylindrical symmetry propagating in a homogeneous medium with a refractive index n. It is well known that the curve $w(z)$ is the profile of a hyperboloid of revolution which may be called the beam surface. It is also known that a hyperboloid of revolution can be generated by a skew line rotating about the axis as shown in Fig. 1. Let us consider a generating skew line whose intersection with the beam waist plane is oriented at an angle θ with respect to the y axis. The projections of the skew line on the xz and yz planes are, respectively, given by

$$\begin{bmatrix} \xi(z) \\ \eta(z) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{2z}{knw_0} \\ w_0 \end{bmatrix}, \quad (12)$$

w_0 being the beam waist radius and the origin of z being taken at the beam waist. The beam surface is generated by the skew line described by Eq. (12) as θ varies from 0 to 2π. Equation (12) can be written explicitly as

$$\begin{aligned} \xi(z) &= -w_0 \sin\theta + \frac{2z}{knw_0}\cos\theta, \\ \eta(z) &= w_0 \cos\theta + \frac{2z}{knw_0}\sin\theta. \end{aligned} \quad (13)$$

We remark that the quantity

$$n[\dot{\xi}(z)\eta(z) - \xi(z)\dot{\eta}(z)] = \frac{2}{k} \quad (14)$$

is invariant as the beam propagates in free space. It is easy to show that this quantity is also invariant as the beam propagates through any optical system with cylindrical symmetry.

Let us show that when a Gaussian beam is transformed by a thin lens the generating skew rays are transformed according to the laws of geometrical optics. [Since the congruence of the generating skew rays is not normal, however, they do not form a ray bundle in the sense of geometrical optics. [For a definition of normal congruences see, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon Oxford, 1965), p. 126.]] It is sufficient to prove this property for the particular generating skew ray which intersects the lens at a point

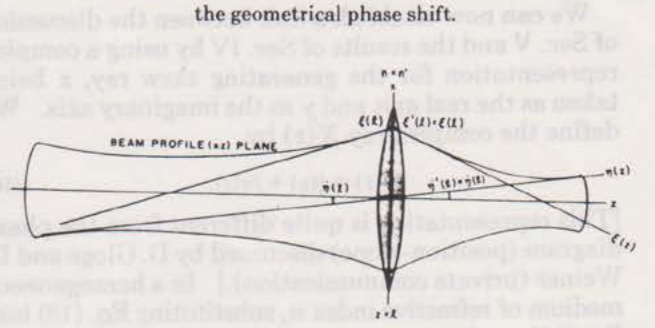

Fig. 2. This figure shows how one can trace the refraction of a Gaussian beam by a lens by tracing two paraxial rays using conventional procedures.

in the xz plane. Its projection $\xi(z)$ on the xz plane is tangent to the beam profile and it is refracted as the beam profile itself, according to the laws of geometrical optics (see Fig. 2). Its projection $\eta(z)$ on the xz plane is equal to zero at the lens plane, $z = l$. Then Eq. (14) shows that

$$n\dot{\xi}(l)\eta(l) = n'\dot{\xi}'(l)\eta'(l), \quad (15)$$

and since $\xi(l) = \xi'(l)$, we have $n\eta(l) = n'\eta'(l)$. Consequently, $\eta(l)$ also obeys the laws of geometrical optics [Fig. 2 is drawn for $n = n'$; in that case, the ray $\eta(z)$, going through the lens center, is a straight line]. This result is readily generalized to the case of an arbitrary orthogonal lenslike medium by substituting for a short section of such a medium a thin lens and a short section of homogeneous medium. The left-hand side of Eq. (14) is still an invariant, n being in general a function of z. The generating skew rays, of course, are no longer straight lines, in general.

We may relate the representation discussed above to the ray packet representation of Gaussian modes[4–6] by considering the projection of all the generating skew rays on a meridional plane. Their envelope is the beam profile. This result was obtained in a slightly different form by Bykov and Vainshtein,[4] Kahn,[5] and Steier.[6]

Let us further remark that there are two generating symmetrical skew rays going through a given point on the beam surface. Considering the rotation about the axis of a point of the generating skew lines, one of them may be said to be clockwise (+) and the other counterclockwise (−). Because of the symmetry, the beam profile at the point considered (i.e., the intersection between the beam surface and the meridional plane which contains that point) is tangent to the bisectrix of the two generating skew rays. Since this property is preserved on projection, we conclude that the bisectrix of any two projected skew lines is tangent to the projected beam profile at the intersecting point. The intersection of this bisectrix with the axis is consequently the center of curvature of the wave front. This is, in substance, the result obtained by Steier.[6]

VI. Complex Representation of the Skew Ray

We can now establish a link between the discussion of Sec. V and the results of Sec. IV by using a complex representation for the generating skew ray, x being taken as the real axis and y as the imaginary axis. We define the complex ray $X(z)$ by

$$X(z) = \xi(z) + j\eta(\xi). \tag{16}$$

[This representation is quite different from the phase diagram (position-slope) discussed by D. Gloge and D. Weiner (private communication).] In a homogeneous medium of refractive index n, substituting Eq. (13) into Eq. (16), we have

$$X(z) = \frac{2\exp(j\theta)}{kw_0 n}\left(z + \frac{jknw_0^2}{2}\right) = \frac{2\exp(j\theta)}{kw_0 n}\, q. \tag{17}$$

From Eq. (5) the on-axis phase shift is

$$-\Phi = \text{phase } X(z) = \theta + \text{phase of } q(z). \tag{18}$$

This expression shows that the choice of θ simply determines the phase reference.

With this complex representation, the left-hand side of Eq. (14) is proportional to the equivalent power $\frac{1}{4}(VI^* + IV^*)$ in the electric circuit equivalence[8,1] $V \to X(z), I \to jn(z)\dot{X}(z)$.

VI. Beam Tracing

In the previous sections we have shown that a fundamental Gaussian beam can be represented by two rays $\xi(z)$ and $\eta(z)$ which obey the laws of geometrical optics. The beam profile is $w(z) = [\xi(z)^2 + \eta(z)^2]^{1/2}$ and the on-axis phase is $\Phi = -\tan^{-1}\eta(z)/\xi(z)$. This representation provides a method for tracing the beam profile and its on-axis phase as it propagates through an optical system.

If a Gaussian beam is known by its beam waist radius (w_0) and position, we may take $\xi(z)$ as a line parallel to the axis at a distance w_0 from it, and $\eta(z)$ as a line crossing the axis at the beam waist plane with an angle equal to the far-field angle $\lambda/\pi w_0$. Neither of these two rays, after transformation by an optical system according to the laws of geometrical optics, is any longer parallel to the optical axis, in general. Figure 3 gives a procedure, based on descriptive geometry, to find the beam waist in that case ($\theta \neq 0$, mod$\pi/2$) with the help of an auxiliary projection on the xy plane. In this xy plane, the minimum distance between the skew line and the z axis (beam waist) is given by the perpendicular

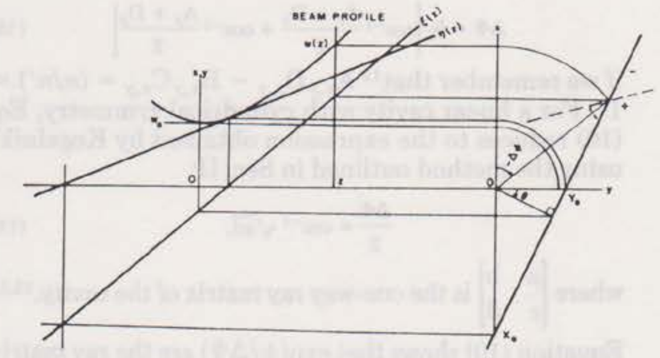

Fig. 3. Procedure to determine the waist of the beam using an auxiliary projection on the xy plane.

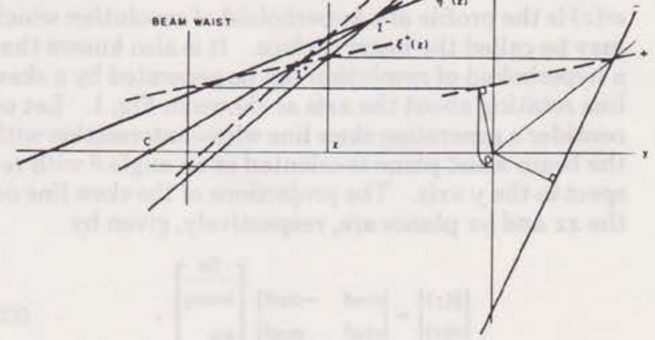

Fig. 4. Determination of the wave front curvature center C. Here the projections of two skew rays need to be traced.

drawn from the origin to the projected skew line. Their intersection is easily drawn back into the xz and yz planes.

The phase front center of curvature C at any plane can be obtained by tracing the other generating skew ray passing through the same point. Its xy projection, shown as a dashed line in Fig. 3, is symmetrical to the given projection (plain line) with respect to a line drawn from the origin. Since C is on the bisectrix of the two skew rays, it lies in the plane that they form and it is, by definition, on-axis. C is consequently aligned with the intersections of the skew rays with an arbitrary meridional plane. A convenient choice for this meridional plane is the plane which makes an angle of 45° with x. The intersections I^- and I^+ with that plane are given by the intersections of the projected lines ($\xi^+ = \eta^+$ and $\xi^- = \eta^-$) and are readily obtained, as shown in Fig. 4.

Another procedure, consisting of constructing the lateral foci and using the method of Deschamps and Mast,[8] is probably preferable if the phase front centers have to be known at many planes. The lateral foci can be obtained by noting that the phase front curvature center is easily obtained at the plane where $\xi(z)$ and $\eta(z)$ intersect. $\xi(z)$ and $\eta(z)$ are indeed conjugate rays for a fictitious mirror situated at that plane with a radius equal to the phase front surface radius.

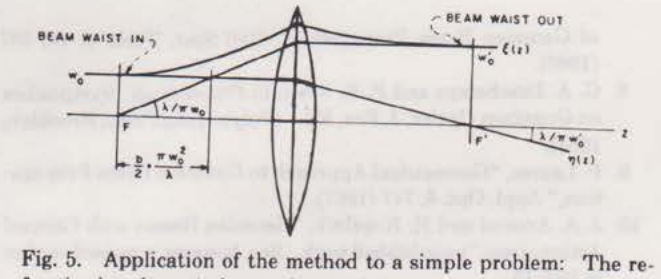

Fig. 5. Application of the method to a simple problem: The refraction by a lens of a beam whose waist coincides with the focal plane of the lens. The tracing readily shows that the beam waist after traversing the lens is located at the image focal plane.

VIII. Application

The construction method outlined above is applied in this section to a simple problem. Let us find the output beam waist radius and position assumed by a beam which has passed through a lens, when the input beam waist is at the object focal point. The tracing readily shows that the beam waist is at the image focal point of the lens as illustrated in Fig. 5.

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Appendix

The purpose of this Appendix is twofold. First, to explain perhaps more clearly the principle of the complex ray representation of Gaussian beams. Second, to discuss more recent works (1968–1984) on the subject. A review, however, is not intended.

Let us suppose that we have been able to find the solution of a partial differential equation describing the propagation of waves in some medium and that this solution depends on arbitrary real parameters. Then, physically different solutions are found by giving complex values to these parameters, provided the complex values that result for the wave function can be given a physical meaning. More specifically, let us consider a time-harmonic field. As usual an $\exp(j\omega t)$ factor is suppressed with the understanding that, whenever the real field is wanted, the complex field is to be multiplied by that $\exp(j\omega t)$ factor and the real part is to be taken. Let us further assume that the medium is invariant for an arbitrary translation z_0 along some propagation axis z. In other words, the medium is supposed to be homogeneous along z. This is the case for uniform waveguides. Then, for obvious physical reasons, if $E(z)$ is a known solution, $E(z - z_0)$, where z_0 represents a real displacement, is also a solution. New physics is obtained, however, and somewhat unexpected solutions discovered, if complex values of the form $a + jb$ are given to z_0. Similarly, if we know a solution $E(r,\phi,z)$ of a rotationally invariant system (e.g., a circular waveguide), a new solution of the form $E(r,\phi - \phi_0,z)$, where ϕ_0 may be complex valued is easily obtained. Alternatively, we may restore arbitrary time dependence and factor out instead an $\exp(-jk_x x)$ dependence on x, that is, keep a constant spatial frequency in place of a time angular frequency.

Let us now go back to time-harmonic fields and to the simplest solution of the scalar wave equation, the one corresponding to a radiating point source. In acoustics, this point source may be a pulsating sphere whose diameter is very small compared to wavelength. Then the classical solution is $E(x,y,z) = \exp(-jkr)/r$, where $r = (x^2 + y^2 + z^2)^{1/2}$. Because the medium is supposed to be homogeneous, clearly $E(x,y,z - z_0)$ is also a solution. If we set $z_0 = -j\zeta$, $\zeta > 0$ and apply the paraxial wave approximation $x^2 + y^2 \ll z^2$, we readily find that $E(x,y,z - z_0)$ describes the field of a Gaussian beam, that is, a field whose amplitude decays as a function of the distance from the z axis according to an $\exp(-x^2)$ type law. This observation was first published by Arnaud as a footnote in Ref. 13 in 1969. A similar remark was made by Deschamps in 1971 for a radiating dipole.[14] In electromagnetism, the dipole is indeed the most elementary radiating system. But in many problems of weak guidance, the scalar approximation is appropriate. The generation of Gaussian beams with general astigmatism was obtained by Arnaud and Kogelnik[15] who introduced the complex rotation alluded to before. The generation of Gaussian beams by complex displacements is often used to treat weakly diverging antenna problems (see, for example, Ref. 16). Most authors in this field refer to the better known paper by Deschamps.[14] Note that the use of complex rays beyond the paraxial approximation is not exempt from difficulties, as discussed in Ref. 16.

Finally, let us note that free-space electromagnetic fields with arbitrary time and space dependences can be generalized along the same concepts if one considers the complex field $G = E - jB$, where E and B are the real electric and magnetic fields. New solutions are obtained if we introduce complex space–time shifts, rotations, boosts, etc. (see Ref. 17 for an interesting but mathematically involved paper).

The complex ray representation of Gaussian beams presented in this paper rests on somewhat different principles than the one just discussed. Indeed, the medium is not supposed to possess any obvious symmetry such as translational or rotational invariance. However, the medium is supposed to exhibit at most quadratic variations of the refractive index as a function of the transverse coordinates (more precisely, one must assume that the z component of the local wave vector $\mathbf{k}$ is at most a quadratic function of x, y and k_x,k_y. For isotropic media, the latter condition implies the paraxial approximation). Now, in such a condition, it is easy to see that the field of a ray pencil can be expressed in terms of a paraxial ray of equation $x = X(z)$, even if the explicit solution of the ray equation is not known. More precisely, the field of the ray pencil is easily expressed (see the main text of this paper) in terms of $(dX/dz)/X$. But $X(z)$ is the solution of a second-order differential equation; it therefore depends on two arbitrary constants. One of these constants disappears when the ratio dX/dz and X is taken, but we are still left with one arbitrary constant. For ordinary rays, $X(z)$ and therefore the arbitrary constant must be real quantities. At the ray pencil center, $X = 0$ and the associated field is therefore infinite. Gaussian beams are obtained, however, if complex values are given to the arbitrary constant. Then the field is finite everywhere. This approach was first published by Arnaud[12] in 1969 and generalized to many optical systems[18,19] and Gaussian pulses.[20]

The main point of this paper is that Gaussian beams can be traced simply by tracing two real paraxial rays, the real and imaginary parts of the complex ray $X(z)$. This beam tracing procedure has been reconsidered recently in this Journal in an interesting paper by Herloski *et al.*[21]

As we indicated earlier, this Appendix is not intended to be a review of the subject. The early work by Kravtsov,[22] however, must be mentioned.

References

1. H. W. Kogelnik, "On the Propagation of Gaussian Beams of Light Through Lenslike Media Including Those with a Loss or Gain Variation," Appl. Opt. **4**, 1562 (1965).
2. H. W. Kognelnik and T. Li, "Laser Beams and Resonators," Appl. Opt. **5**, 1550 (1966).
3. H. W. Kogelnik, "Imaging of Optical Modes—Resonators with Internal Lenses," Bell Syst. Tech. J. **44**, 455 (1965).
4. V. P. Bykov and L. A. Vainshtein, "Geometric Optics of Open Resonators," Sov. Phys. JETP **20**, 338 (1965).
5. W. K. Kahn, "Geometric Optical Derivation of Formula for the Variation of the Spot Size in a Spherical Mirror Resonator," Appl. Opt. **4**, 758 (1965).
6. W. H. Steier, "The Ray Packet Equivalent of a Gaussian Light Beam," Appl. Opt. **5**, 1229 (1966).
7. S. A. Collins, Jr., "Analysis of Optical Resonators Involving Focusing Elements," Appl. Opt. **3**, 1263 (1964); T. Li, "Dual Forms of the Gaussian Beam Chart," Appl. Opt. **3**, 1315 (1964); J. P. Gordon, "A Circle Diagram for Optical Resonator," Bell Syst. Tech. J. **43**, 1826 (1964); T. S. Chu, "Geometrical Representation of Gaussian Beam Propagation," Bell Syst. Tech. J. **45**, 287 (1966).
8. G. A. Deschamps and P. E. Mast, in *Proceedings, Symposium on Quantum Optics*, J. Fox, Ed. (Polytechnic Press, Brooklyn, 1964).
9. P. Laures, "Geometrical Approach to Gaussian Beam Propagation," Appl. Opt. **6**, 747 (1967).
10. J. A. Arnaud and H. Kogelnik, "Gaussian Beams with General Astigmatism," unpublished work. See, however, a revised version in Ref. 15.
11. R. K. Luneburg, *Mathematical Theory of Optics* (U. California Press, Berkeley, 1964), Chap. 4.
12. J. A. Arnaud, "Degenerate Optical Cavities," Appl. Opt. **8**, 189 (1969). The relevant footnote is on p. 190; Note that this paper was submitted in May 1968.
13. J. A. Arnaud, "Degenerate Optical Cavities. 2: Effect of Misalignments," Appl. Opt. **8**, 1909 (1969). The principle of complex coordinate shifts is stated in the footnote on p. 1910.
14. G. A. Deschamps, "The Gaussian Beam as a Bundle of Complex Rays," Electron. Lett. **7**, 684 (1971).
15. J. A. Arnaud and H. Kogelnik, "Gaussian Light Beams with General Astigmatism," Appl. Opt., **8**, 1687 (1969).
16. I. Montrosset and R. Orta, "La méthode des rayons complexes," Ann. Télécommun. **38**, 1 (1983).
17. D. Weingarten, "Complex Symmetries of Electrodynamics," Ann. Phys. **76**, 510 (1973).
18. J. Arnaud, *Beam and Fiber Optics* (Academic, New York, 1976), Chap. 2.
19. N. Aoyagi and S. Yamaguchi, "Generalized Fresnel Transformations and Their Properties," Jpn. J. Appl. Phys. **12**, 1343 (1973).
20. J. A. Arnaud, "A Theory of Gaussian Pulse Propagation," Opt. Quantum Electron. **16**, 125 (1984).
21. R. Herloski, S. Marshall, and R. Antos, "Gaussian Beam Ray-Equivalent Modeling and Optical Design," Appl. Opt. **22**, 1168 (1983).
22. Y. Kravtsov, "Complex Rays and Complex Caustics," paper presented at the Fourth All-Union Symposium on Diffraction of Waves, Moscow, U.S.S.R. (Nauka, Moscow, 1967).