Optical and Quantum Electronics **10** *(1978) 53–60*

Ray theory of the impulse response of randomly bent multimode fibres

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Received 2 August 1977

A ray theory based on the time-independent Fokker–Planck equation and the integration of time along ray trajectories provides analytical expressions for the average arrival time and spread of optical pulses propagating in randomly distorted, multimode, optical fibres. A clear physical picture emerges from the theory. The analytical expressions obtained for $\langle t \rangle$ and $\langle t^2 \rangle$ coincide with the ones obtained by Olshansky from coupled-mode theory. The $\langle t^3 \rangle$ and $\langle t^4 \rangle$ moments of the impulse response are also calculated. Simple closed-form formulae are given for the step-index slab. The coupling between all modes is effectively taken into account in our ray theory.

1. Introduction

It is well known that random perturbations of the geometry of multimode optical waveguides reduce their temporal dispersion [1–4]. In the absence of perturbation, the width $\sigma(L)$ of the impulse response is proportional to the length L of the fibre. However geometrical imperfections induce fluctuations of the ray trajectories (or, equivalently, induce coupling between the guided modes). As a result, the width of the impulse response increases only in proportion to the square root of L, for large values of L. One of the most complete theories of that effect to date is probably that of Olshansky [5]. Olshansky's theory is based on a perturbation solution of the time-dependent Fokker–Planck equation, the latter being obtained from Marcuse's coupled mode theory [3–4].

In the present paper we show that the same results can be obtained by using the time-independent Fokker–Planck equation, derived from ray-optics in [6] and [7], and integrating the time along ray trajectories. Thus, a fully ray-optics theory is obtained. Our approach provides a clearer physical picture of the processes involved than modal and perturbation techniques. In Sections 2–6 we derive a number of general results based on probability theory and, in Section 7, we illustrate these results by applying them to the step-index slab. Readers, if they wish, may read Section 7 first where simple closed-form results are given.

2. The time-dependent Fokker–Planck equation

A time-dependent Fokker–Planck equation of the form

$$\partial R(m, z, t)/\partial z + \tau(m)\partial R(m, z, t)/\partial t = \mathfrak{L}\,(m, \partial/\partial m)R(m, z, t) \tag{1}$$

has been derived by Marcuse [3, 4] from coupled-mode theory, under the assumption that the guide deformations are stationary. In Equation 1 R denotes the power in mode m at the axial coordinate z and time t. $\tau(m)$ denotes the reciprocal of the group velocity of mode m. $\mathfrak{L}$ is a linear operator, independent of z, proportional to the strength of the deformation. The Fokker–Planck equation in Equation 1 can be

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obtained as well from a ray theory [6, 7]. In a ray theory, the mode number m is replaced by a vector s that specifies a point in phase-space (in short s is called a phase vector). In the special case of step-index slabs for example, m should be replaced by the angle that the ray makes with the guide axis. Note that, within the axiomatics of space-time ray optics, optical powers, rather than optical fields, add linearly.

Olshansky [5] was able to obtain analytical expressions for the average time of arrival of a pulse using a first-order perturbation of Equation 1, and for the width of the output pulse using a second order perturbation of Equation 1. The approach described in the remainder of this paper differs vastly from that of Olshansky in concept and in the details of the calculation. Yet, it provides the same answers. Our alternative formulation is sketched in the next section.

3. An alternative formulation for impulse response analysis

Consider first a time-independent source and define the probability density $P(\mathrm{s}, z)$ where the phase-vector is s at z. The vector $\mathrm{s} = (x, \mathrm{d}x/\mathrm{d}z, y, \mathrm{d}y/\mathrm{d}z)$ denotes the position and slope of the ray, at some z. That is $P(\mathrm{s}, z)\,\mathrm{d}x\,\mathrm{d}(\mathrm{d}x/\mathrm{d}z)\ldots$ denotes the probability that the ray has a position between x and $x + \mathrm{d}x$, slope between $\mathrm{d}x/\mathrm{d}z$ and $\mathrm{d}x/\mathrm{d}z + \mathrm{d}(\mathrm{d}x/\mathrm{d}z)\ldots$, at fixed z. The probability is defined over an ensemble of fibres. (The word 'density' is sometimes omitted). Under the usual assumptions, this probability obeys the time-independent Fokker–Planck equation

$$\partial P(\mathrm{s}, z)/\partial z = \mathfrak{L}(\mathrm{s}, \partial/\partial \mathrm{s})P(\mathrm{s}, z) \tag{2}$$

where the linear operator $\mathfrak{L}$ is the same as in Equation 1. Without loss of generality, we assume that $\mathfrak{L}$ is self-adjoint. Then, the forward and backward Fokker–Planck equations coincide.

The transit time of a pulse along a ray $\mathrm{s} = \mathrm{s}(z)$ is

$$t(z) = \int_0^z \tau[\mathrm{s}(z)]\,\mathrm{d}z \tag{3}$$

where τ is the time of propagation of a pulse along the ray over a ray period, divided by the ray period. This time can be evaluated by integrating $\mathrm{d}l/u$ along the ray, where $\mathrm{d}l$ denotes the elementary ray length and u the local group velocity. The set of Equations 2 and 3 is fully equivalent to Equation 1. This, incidentally, proves that, within space-time geometrical optics, the time dependent Fokker–Planck equation does not contain diffusion terms of the form $\partial^2P/\partial t\partial \mathrm{s}$ or $\partial^2P/\partial t^2$, except perhaps in singular cases (See [6]).

In subsequent sections we will evaluate the average time of arrival of a pulse

$$\langle t(L)\rangle = \int_0^L \langle \tau[\mathrm{s}(z)]\rangle\,\mathrm{d}z \tag{4}$$

and the average square of the time of arrival

$$\langle t^2(L)\rangle = 2\int_0^L \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2\,\langle \tau[\mathrm{s}(z_1)]\tau[\mathrm{s}(z_2)]\rangle \tag{5}$$

which the r.m.s. impulse response width squared $\sigma^2 = \langle t^2\rangle - \langle t\rangle^2$ follows.

In Equations 4 and 5, L denotes the length of the fibre and $\langle\ \rangle$ denotes an ensemble average. The latter concept will be made more precise later.

4. Modal solutions

In step-index slabs, rays are transmitted, in principle without loss, if $|\theta| \leqslant \sqrt{(2\Delta)}$ where Δ denotes the relative change of index. The ray is absorbed if $|\theta| > \sqrt{(2\Delta)}$. $|\theta|$ denotes the absolute value of the ray angle to the curved fibre axis (Fig. 1).

More generally, we assume that a ray is transmitted if the phase vector s is within some specified boundary ϵ, and eliminated otherwise. Let T_{0z} denote the event that the ray has been transmitted up to z, that is, s has remained within ϵ all the way from 0 to z. Further, let $P(\mathrm{s}, z; T_{0z})\,\mathrm{d}\mathrm{s}$ denote the prob-

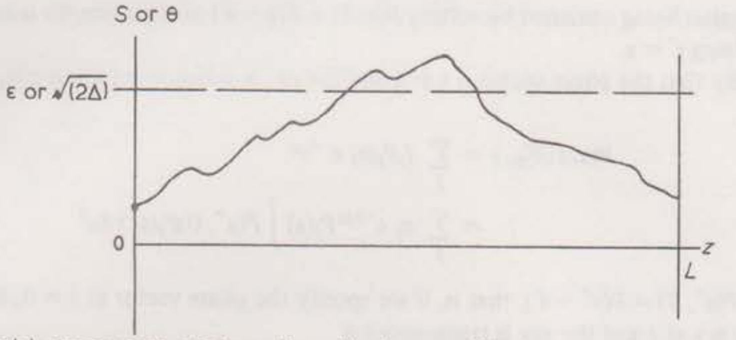

Figure 1 Variation of $\theta(z)$, the ray angle, at intersection with the curved fibre axis as a function of z. The ray sample shown in the figure is not transmitted because θ exceeds $\sqrt{(2\Delta)}$ somewhere between $z = 0$ and $z = L$.

ability that a ray is between s and s + ds at z and has been transmitted up to z. $P(\mathrm{s}, z; T_{0z})$ is a solution of Equation 2 with the boundary condition: $P = 0$ for s on ϵ. To obtain $P(\mathrm{s}, z; T_{0z})$, it is convenient to first define steady-state probability distributions $P_j(\mathrm{s})$ and losses λ_j from the eigenvalue equation

$$\begin{cases} -\lambda_j P_j(\mathrm{s}) = \mathfrak{L}(\mathrm{s}, \partial/\partial \mathrm{s})P_j(\mathrm{s});\ j = 1, 2\ldots \\ P_j(\mathrm{s}) = 0 \ \text{ on } \epsilon. \end{cases} \tag{6}$$

In order to interpret $P_j(\mathrm{s})$ as a probability distribution, it must be normalized by

$$\int P_j(\mathrm{s})\,\mathrm{d}\mathrm{s} = 1 \tag{7}$$

the integral in Equation 7 being taken within the specified boundary ϵ. Note that our normalization in Equation 7 differs from that used in [5]. The well-known orthogonality property of the P_j is denoted

$$\int P_j(\mathrm{s})P_k(\mathrm{s})\,\mathrm{d}\mathrm{s} = \delta_{jk}a_j^{-1} \tag{8}$$

where δ_{jk} is the usual Kronecker symbol, and the a_j are numbers that can be evaluated once the P_j have been obtained from Equations 6 and 7.

Any ray probability distribution $P(\mathrm{s}, 0)$ at $z = 0$ can be written as a series $P_j(\mathrm{s})$

$$P(\mathrm{s}, 0) = \sum_j I_j P_j(\mathrm{s}). \tag{9a}$$

Using the orthogonality (Equation 8) of the P_j and postulating completeness for the class of functions $P(\mathrm{s}, 0)$ that satisfy the same boundary condition as the P_j, the coefficients I_j are given by

$$I_j = a_j \int P(\mathrm{s}, 0)P_j(\mathrm{s})\,\mathrm{d}\mathrm{s}; \tag{9b}$$

$$\sum_j a_j P_j(\mathrm{s})P_j(\mathrm{s}') = \delta(\mathrm{s} - \mathrm{s}') \tag{9c}$$

and, since $P(\mathrm{s}, 0)$ is normalized to unity, we have

$$\sum_j I_j = 1 \tag{9d}$$

$$\sum_j a_j P_j(\mathrm{s}) = 1 \tag{9e}$$

The latter expression being obtained by setting $P(\mathbf{s}, 0) = \delta(\mathbf{s} - \mathbf{s}')$ in Equation 9b using Equation 9d, and subsequently setting $\mathbf{s}' = \mathbf{s}$.

The probability that the phase vector is $\mathbf{s}$ at z and the ray is transmitted up to z is, using Equations 2, 6 and 9

$$P(\mathbf{s}, z; T_{0z}) = \sum_j I_j P_j(\mathbf{s})\, \mathrm{e}^{-\lambda_j z}$$

$$= \sum_j a_j\, \mathrm{e}^{-\lambda_j z} P_j(\mathbf{s}) \int P(\mathbf{s}'', 0) P_j(\mathbf{s}'')\, \mathrm{d}\mathbf{s}'' \qquad (10)$$

In particular, if $P(\mathbf{s}'', 0) = \delta(\mathbf{s}'' - \mathbf{s}')$, that is, if we specify the phase vector at $z = 0$, the probability that the phase vector is $\mathbf{s}$ at z and the ray is transmitted is

$$P(\mathbf{s}, z; T_{0z}|\mathbf{s}', 0) = \sum_j a_j P_j(\mathbf{s}) P_j(\mathbf{s}')\, \mathrm{e}^{-\lambda_j z}. \qquad (11)$$

If we integrate this expression over $\mathbf{s}$ we obtain the probability f that a ray is transmitted up to z, given that the phase vector equals $\mathbf{s}'$ at $z = 0$

$$f(T_{0z}|\mathbf{s}', 0) = \sum_j a_j P_j(\mathbf{s}')\, \mathrm{e}^{-\lambda_j z}. \qquad (12)$$

Finally, the probability that a ray is transmitted from 0 to L is, from Equation 10

$$f(T_{0L}) = \int P(\mathbf{s}, L; T_{0L})\, \mathrm{d}\mathbf{s} = \sum_j I_j\, \mathrm{e}^{-\lambda_j L}. \qquad (13)$$

5. Average arrival time

To evaluate the average arrival time under the assumption that all the rays are excited at the input of the fibre at time $t = 0$ by a δ-function, we need the probability distribution $P(\mathbf{s}, z|T_{0L})$ of $\mathbf{s}$ at z conditional on the fact that the ray will reach the end of the fibre (of length L). This conditional probability is

$$P(\mathbf{s}, z|T_{0L}) = P(\mathbf{s}, z; T_{0L})/f(T_{0L}). \qquad (14)$$

$P(\mathbf{s}, z; T_{0L})$ in Equation 14 is equal to the three-dimensional probability $P(\mathbf{s}, z; T_{0z}; T_{zL})$, or, because $\mathbf{s}(z)$ is a Markov process

$$P(\mathbf{s}, z; T_{0L}) = P(\mathbf{s}, z; T_{0z}) f(T_{zL}|\mathbf{s}, z). \qquad (15)$$

Thus, using the expressions in Equations 10, 12 and 13, the conditional probability in Equation 14 is finally obtained as

$$P(\mathbf{s}, z|T_{0L}) = P(\mathbf{s}, z; T_{0z}) f(T_{zL}|\mathbf{s}, z)/f(T_{0L})$$

$$= \frac{\sum_j I_j P_j(\mathbf{s})\, \mathrm{e}^{-\lambda_j z} \sum_m a_m P_m(\mathbf{s})\, \mathrm{e}^{-\lambda_m (L-z)}}{\sum_n I_n\, \mathrm{e}^{-\lambda_n L}}. \qquad (16)$$

The average arrival time given in Equation 4 is, more explicitly

$$\langle t(L)\rangle = \int_0^L \mathrm{d}z \int \mathrm{d}\mathbf{s}\, \tau(\mathbf{s}) P(\mathbf{s}, z|T_{0L}). \qquad (17)$$

Using in Equation 17 the expression given in Equation 16, a straightforward integration over z gives

$$\langle t(L)\rangle = \frac{\sum_m a_m\, \mathrm{e}^{-\lambda_m L}\left[I_m \mu_{mm} L + \sum_j I_j \mu_{mj} E_{mj}(L)\right]}{\sum_n I_n\, \mathrm{e}^{-\lambda_n L}} \qquad (18\mathrm{a})$$

where we have defined

$$\mu_{mj} = \int \tau(\mathbf{s}) P_m(\mathbf{s}) P_j(\mathbf{s})\, \mathrm{d}\mathbf{s}$$

$$E_{mj}(L) = (\mathrm{e}^{\lambda_{mj} L} - 1)/\lambda_{mj}$$

$$\lambda_{ab} \equiv \lambda_a - \lambda_b. \qquad (18\mathrm{b})$$

This result coincides with that given by Olshansky in [5] if we take into account the differences in normalization of the P_j.

For small values of L, the expression in Equation 18 can be shown to reduce to

$$\langle t(L)\rangle = L \int \tau(\mathbf{s}) P(\mathbf{s}, 0)\, \mathrm{d}\mathbf{s}; \quad \lambda_1 L \ll 1. \qquad (19)$$

This result is physically obvious because the ray distribution $P(\mathbf{s}, 0)$ at the input of the fibre remains essentially undisturbed over a short length of fibre.

The case of large lengths L is more interesting. Because the λ_j increase rapidly with j, the main contribution to $\langle t(L)\rangle$ comes from the first stationary distribution $j = 1$, provided I_1 is not small in comparison to the other coefficients. We thus obtain the result for $I_1 \neq 0$

$$\langle t(L)\rangle \sim a_1 \mu_{11} L = L \int \tau(\mathbf{s}) P_1^2(\mathbf{s})\, \mathrm{d}\mathbf{s} \Big/ \int P_1^2(\mathbf{s})\, \mathrm{d}\mathbf{s}; \quad \lambda_1 L \gg 1 \qquad (20)$$

which is independent of the excitation conditions.

In both cases (small L and large L), the average time of arrival is proportional to length, but the coefficients (average velocities) are quite different in these two limits.

6. Average impulse response width

The second moment $\langle t^2(L)\rangle$ defined in Equation 5 can be evaluated exactly by similar methods. The result again coincides with that of Olshansky [5].

The details are given in the Appendix. The asymptotic expression for the variance $\sigma^2 = \langle t^2\rangle - \langle t\rangle^2$ is

$$\sigma^2 \sim 2L \sum_{m=2}^{\infty} \mu_{1m}^2 a_1 a_m/\lambda_{m1}; \quad \lambda_1 L \gg 1. \qquad (21\mathrm{a})$$

All the higher order moments can also be evaluated exactly because we know the temporal distribution of the Markov random function $\mathbf{s}(z)|T_{0L}$ in Equation 3 (we have to consider only the transmitted rays). After simple calculations, we obtain

$$\sigma^{(3)} \equiv \langle\{[t(L) - \langle t(L)\rangle]/\sqrt{L}\}^3\rangle \to 0, \quad L \to \infty \qquad (21\mathrm{b})$$

and moreover

$$\sigma^{(4)} \equiv \langle\{[t(L) - \langle t(L)\rangle]/\sqrt{L}\}^4\rangle \to 3\sigma^4, \quad L \to \infty. \qquad (21\mathrm{c})$$

We know that for a Gaussian random variable, we have $\sigma^{(4)} = 3\sigma^4$. The results in Equation 21 suggest that the response tends to be Gaussian for large lengths. This conclusion was reached by Personick [1] for two modes, and by Marcuse [3] from second-order perturbation theory.

7. Application to the step-index slab

The very general results given in the previous sections are now applied to the step-index slab. Closed-form results are exceedingly simple to obtain when the correlation of the fibre axis curvature is microscopic (white spectrum). Since the main purpose of this section is to illustrate the previous results we shall make that assumption.

Consider first an homogeneous medium, and let $C(z)$ denote the local curvature of the curved z axis. With respect to that curved axis, the ray equation is

$$\theta(z) = \theta(0) + \int_0^z C(z')\,dz' \tag{22}$$

where $\theta(z) = dx(z)/dz$ denotes the angle of the ray to the z axis. Let us assume that $C(z)$ is a stationary process of zero mean and microscopic correlation. We obtain with the usual transformation

$$\langle[\theta(z)-\theta_0]^2\rangle = \left\langle \int_0^z \int_0^z C(z')C(z'')\,dz'\,dz'' \right\rangle = \gamma z \tag{23}$$

where $\theta_0 = \theta(0)$ and

$$\langle C(z')C(z'')\rangle = \gamma\delta(z'-z''). \tag{24}$$

In Equation 24 $\delta(z)$ denotes Dirac's δ-function, and γ is the power spectral density of the process $C(z)$ (defined over the $-\infty, +\infty$ domain of angular spatial frequencies).

If $C(z)$ is a Gaussian process, it follows from the linearity of Equation 22 that $\theta(z)$ is also a Gaussian process. However $\theta(z)$ tends to be Gaussian for large z even if $C(z)$ is non-Gaussian, as a consequence of the central-limit theorem. Thus, the probability distribution of θ is always, for large z, given by

$$P(\theta, z|\theta_0) = (2\pi\gamma z)^{-1/2} \exp[-\tfrac{1}{2}(\theta-\theta_0)^2/\gamma z] \tag{25}$$

which is recognized as the Green function of the heat-diffusion-type equation

$$\partial P/\partial z = (\gamma/2)\partial^2 P/\partial\theta^2. \tag{26}$$

Note that our derivation provides the Fokker–Planck equation Equation 26 without the need of invoking the standard Fokker–Planck theory.

We now wish to account for the effect of the slab boundaries by specifying that $P = 0$ when $|\theta| = \sqrt{(2\Delta)}$, where $\Delta = \Delta n/n$ is the relative change of refractive index, for all z. (Note that we have been able to ignore the sign reversal of dx/dz at the slab boundaries because of our assumption that the curvature has microscopic correlation, that is, has a white spectrum).

The normalized stationary solutions of Equation 26 with the boundary condition $P(\pm\sqrt{(2\Delta)}, z) = 0$ is

$$P_j(\theta, z) = P_j(\theta)\,e^{-\lambda_j z} \tag{27a}$$

where

$$P_j(\theta) = [(2j-1)\pi/4\sqrt{(2\Delta)}] \cos[(2j-1)\tfrac{\pi}{2}\theta/\sqrt{(2\Delta)}](-1)^{j+1} \tag{27b}$$

$$\lambda_j = [(2j-1)\pi/4]^2\gamma/\Delta$$

$$a_j^{-1} = (2j-1)^2\pi^2/[16\sqrt{(2\Delta)}]; \quad j = 1, 2 \ldots$$

The lowest steady-state loss is therefore*

$$\lambda_1 = (\pi/4)^2\gamma/\Delta = 0.617\gamma/\Delta \tag{28}$$

$$\alpha = 2.65\gamma/\Delta \text{ dB/unit length.}$$

The steady-state radiation pattern is (apart from a possible effect of refraction at the fibre tip)

$$I(\theta) = \cos\left[\frac{\pi}{2}\theta \Big/ \sqrt{(2\Delta)}\right] \tag{29}$$

to within a constant factor.

For step-index fibres, material dispersion is a negligible effect. We may therefore take for $\tau(\theta)$ the well-known expression

$$\tau(\theta) = (n_0/c)/\cos\theta \approx (n_0/c)(1+\theta^2/2) \tag{30}$$

where c/n_0 denotes the velocity of axial pulses. The average arrival time for large L is obtained by substituting Equation 30 in Equation 20 and integrating over θ. Omitting the term n_0L/c, we obtain

*For comparison, for a square-law fibre, we have $\lambda_1 = 0.72\gamma/\Delta$.

$$\langle t(L)\rangle = \left(\frac{1}{3} - \frac{2}{\pi^2}\right) n_0 L\Delta/c \approx 0.13 n_0 L\Delta/c. \tag{31}$$

The square of the r.m.s. impulse response width is, substituting Equation 30 in Equation 21a

$$\sigma^2(L) \approx 2.24 \times 10^{-3}(n_0^2\Delta^3 L/\gamma c^2). \tag{32}$$

Thus, if we define a transmission capacity improvement factor $R = \sigma/\sigma_0$ as the ratio of $\sigma(L)$ in Equation 32 and the r.m.s. impulse width σ_0 in the absence of perturbation assuming that all modes are equally excited at the input

$$\sigma_0 = 0.217 n_0 L\Delta/c \tag{33}$$

we find, in agreement with Marcuse's theory, that, for large fibre lengths, the product of R^2 and the total loss αL given in Equation 28 is a constant

$$R^2\alpha L = \text{(Transmission capacity improvement)}^2\text{(Total loss)} = 0.132 \text{ dB}. \tag{34}$$

The Olshansky result for a round step-index fibre is

$$R^2\alpha L = 0.75 \text{ dB (see curve } p = 0, \alpha = 100 \text{ in Fig. 6 of [5])}.$$

Note that the Olshansky result is approximate because only the coupling between adjacent modes is taken into account and because, contrary to an assumption made, the modes of the step-index fibre are not degenerate.

8. Conclusion

A general, full ray-optics theory of slightly distorted multimode fibres has been given. For a given nominal index profile we first evaluate ray trajectories in the undistorted fibre and calculate the average time of flight per unit axial length. We subsequently evaluate the diffusion of rays in phase space due to the distortion of the fibre. This diffusion process obeys a time-independent Fokker–Planck equation. The quantities of practical interest are the moments $\langle t\rangle, \langle t^2\rangle \ldots$ of the impulse response. They have been given in terms of the modal solutions of the Fokker–Planck equation. Simple closed form formulae have been obtained for step-index slabs.

Appendix

Derivation of the second moment $\langle t^2(L)\rangle$

The derivation is formally almost the same as for the first moment $\langle t(L)\rangle$ in Section 5. We need the property that the random function $s(z)$ is a Markov process.

We start from Equation 5 where the mean values are still understood to be conditional on the event T_{0L} that the ray will reach the end L of the fibre i.e. we need the conditional two-dimensional probability distribution

$$P(s_1, z_1; s_2, z_2|T_{0L}) = P(s_1, z_1; s_2, z_2; T_{0L})/f(T_{0L}) \tag{A.1}$$

where $f(T_{0L})$ is still given by Equation 13 and $P(s_1, z_1; s_2, z_2; T_{0L})$ is also equal to the four-dimensional probability distribution

$$P(s_1, z_1; s_2, z_2; T_{0L}) = P(s_1, z_1; s_2, z_2; T_{0z_1}; T_{z_1L}) = P(T_{z_1L}|s_1, z_1; s_2, z_2; T_{0z_1})P(s_1, z_1; s_2, z_2; T_{0z_1}). \tag{A.2}$$

Because the function $s(z)$ is of Markov type, the conditional probability distribution in Equation A.2 is

$$P(T_{z_1L}|s_1, z_1; s_2, z_2; T_{0z_1}) = f(T_{z_1L}|s_1, z_1) \tag{A.3}$$

where f is given in Equation 12. The three-dimensional probability distribution in Equation A.2 may now be expressed in terms of a conditional probability defined in Equation 11.

$$P(s_1, z_1; s_2, z_2; T_{0z_1}) = P(s_1, z_1; T_{z_2 z_1} | s_2, z_2) P(s_2, z_2; T_{0z_2}) \tag{A.4}$$

From Equations 11, 12, A.2–A.4 we can write

$$\langle \tau[s(z_1)] \tau[s(z_2)] \rangle = [P(T_{0L})]^{-1} \sum_{jmk} a_j a_m I_k \mu_{jm} \mu_{km} \, e^{-\lambda_j z} \, e^{(\lambda_j - \lambda_m) z_1} \, e^{(\lambda_m - \lambda_k) z_2}. \tag{A.5}$$

For the integration of Equation A.5 over z_1 and z_2, we have different terms

$$\begin{aligned} \langle t^2(L) \rangle = [P(T_{0L})]^{-1} \sum_{jmk} a_j a_m I_k \mu_{jm} \mu_{km} \Bigg[& \left(\frac{e^{-\lambda_k z} - e^{-\lambda_j z}}{\lambda_{jk} \lambda_{mk}} \right)_{j \neq k,\, m \neq k} - \left(\frac{e^{-\lambda_m z} - e^{-\lambda_j z}}{\lambda_{jm} \lambda_{mk}} \right)_{j \neq m,\, m \neq k} \\ & + \left(\frac{z\, e^{-\lambda_m z}}{\lambda_{jm}} - \frac{e^{-\lambda_m z} - e^{-\lambda_j z}}{\lambda_{jm}^2} \right)_{m = k,\, j \neq m} + \left(\frac{e^{-\lambda_k z} - e^{-\lambda_m z}}{\lambda_{mk}^2} - \frac{z\, e^{-\lambda_m z}}{\lambda_{mk}} \right)_{j = m,\, k \neq j} \\ & + \left(\frac{z\, e^{-\lambda_j z}}{\lambda_{mj}} + \frac{e^{-\lambda_m z} - e^{-\lambda_j z}}{\lambda^2_{mj}} \right)_{j = k,\, m \neq j} + \left(\frac{z^2}{2} e^{-\lambda_m z} \right)_{j = m = k} \Bigg]. \end{aligned}$$

$$\lambda_{ab} \equiv \lambda_a - \lambda_b \tag{A.6}$$

When $L \gg 1/\lambda_1$ and $I_1 \neq 0$, we obtain

$$\langle t^2(L) \rangle \sim L^2 a_1^2 \mu_{11}^2 + 2La_1 \left[\sum_m a_m \mu_{1m}^2 \lambda_{m1}^{-1} + a_m \mu_{11} \mu_{1m} \lambda_{m1}^{-1} \right] + 2La_1^2 \sum_m I_m I_1^{-1} \mu_{11} \mu_{m1} \lambda_{m1}^{-1}. \tag{A.7}$$

The variance of the delay time distribution is now deduced from Equations A.7 and 18a since

$$\langle t(L) \rangle \sim La_1 \mu_{11} + \sum_{m = 2, 3 \ldots} \frac{a_m \mu_{1m}}{\lambda_{m1}} + \frac{a_1 \mu_{1m}}{\lambda_{m_1}} \frac{I_m}{I_1} \tag{A.8}$$

and we finally obtain Equation 21.

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