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Ray theory of randomly bent multimode optical fibers

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The complete solution for propagation in randomly bent, circularly symmetric multimode optical fibers is given; the paraxial-ray-optics approximation is used. This ray-optics solution is, in principle, equivalent to the power-coupled-mode equations in the continuum limit. However, none of the assumptions usually made in modal theories is needed in the ray theory. In particular, the coupling between nonadjacent modes is effectively taken into account.

Introduction

Random bending is the most commonly found defect in multimode optical fibers. These bends have the unfavorable effect of reducing the optical power available at the detector, but they also have the favorable effect of reducing pulse spreading. A fairly complete theory of that effect based on Marcuse's power-coupled-mode theory has been given by Olshansky.[1] The ray theory presented in this Letter is significantly simpler than modal theories. Thus, analytical calculations can be performed with fewer approximations. In particular, the coupling between nonadjacent modes need not be neglected.

Our theory is based on the Fokker–Planck formalism discussed, for example, by Stratonovich.[2] The coefficients entering into that Fokker–Planck (FP) equation are expressed as integrals of ray parameters over a ray period for any index profile. For power-law profiles, closed-form expressions can be found. For the sake of clarity, we assume that the fiber-axis curvature has microscopic correlation. However, nonuniform spectra can be handled just as easily, as in the two-dimensional case treated in Refs. 3–5. Finally, it will be shown that the FP equation can be written in a self-adjoint form that makes comparisons with modal theories straightforward.

Ray Equations

Let $n(x,y)$ denote the index profile of the fiber and $C_x(z),C_y(z)$ the curvature of the fiber axis in the x-z and y-z planes, respectively. A ray trajectory is denoted as $x = x(z)$, $y = y(z)$, x and y being measured from the fiber axis. If we further define

$$U(x,y) = 1 - n(x,y)/n(0,0), \tag{1}$$

the paraxial-ray equations are[6]

$$\ddot{x} = \frac{d^2x}{dz^2} = -\frac{\partial U(x,y)}{\partial x} + C_x(z), \tag{2a}$$

$$\ddot{y} = \frac{d^2y}{dz^2} = -\frac{\partial U(x,y)}{\partial y} + C_y(z). \tag{2b}$$

Note, incidentally, that these equations also describe the motion of a mass in a potential $U(x,y)$ subjected to a driving force C_x,C_y, provided that z is changed to t (time). It would be possible, in principle, to trace rays on the basis of Eqs. (1) and (2) along the full length of the fiber for specified initial conditions and curvatures and determine which one of the excited rays is transmitted. This straightforward approach, however, usually involves large numerical errors. It is thus preferable to investigate how the parameters E,μ that characterize the ray motion evolve as a result of the fiber-axis curvature.

For circularly symmetric profiles $n(r) \rightarrow U(r)$, the ray parameters E,μ are defined as

$$E = U(r) + \frac{1}{2}(\dot{x}^2 + \dot{y}^2);\ r^2 = x^2 + y^2, \tag{3a}$$

$$\mu = x\dot{y} - y\dot{x}, \tag{3b}$$

where the dots denote differentiation with respect to z.

In the mechanical analog, E,μ are called, respectively, energy and angular momentum. In the language of wave optics, these quantities are, respectively to within constant factors, the propagation constant of a mode and the azimuthal mode number. The variation of E and μ as a function of z is obtained by straightforward differentiations and use of Eq. (2):

$$dE/dz = \dot{x}C_x + \dot{y}C_y, \tag{4a}$$

$$d\mu/dz = xC_y - yC_x. \tag{4b}$$

In deriving Eq. (4b), the fact that the fiber is circularly symmetric has been used.

For a straight fiber, let us also define

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$$I(E,\mu) = \oint \dot{r} dr; \dot{r}^2 = 2[E - U(r)] - \mu^2/r^2, \quad (5a)$$

which is analogous to the mechanical action. In the language of wave optics, I represents the radial mode number. By differentiation of the expression of I under the integral sign, we find that the ray axial and azimuthal periods are, respectively,

$$Z(E,\mu) = \partial I(E,\mu)/\partial E, \quad (5b)$$

$$-\Phi(E,\mu) = \partial I(E,\mu)/\partial \mu. \quad (5c)$$

Note the relation

$$2Z(E - \overline{U}) = I + \mu\Phi, \quad (5d)$$

where the bar denotes an average over a ray period.

Finally, we define

$$R = \overline{r^2} = Z^{-1}\oint (r^2/\dot{r}) dr. \quad (6a)$$

We shall need the following identity satisfied by R:

$$\partial(RZ)/\partial\mu + \mu\partial Z/\partial E = 0. \quad (6b)$$

The variation of I as a function of z follows from Eqs. (4) and (5):

$$\frac{dI}{dz} = Z\frac{dE}{dz} - \Phi\frac{d\mu}{dz} = (Z\dot{x} + \Phi y)C_x + (Z\dot{y} - \Phi x)C_y. \quad (6c)$$

In the next section, we take C_x, C_y random.

Statistical Theory

Instead of dealing with a single fiber, it is convenient to consider an ensemble of fibers that differ from one another only in curvature law. Thus we take $C_x(z)$ and $C_y(z)$ random. The sign $\langle\ \rangle$ will refer to averages over the ensemble of fibers. In order to preserve circular symmetry and stationarity in the statistical sense, we assume that $C_x(z)$ and $C_y(z)$ have zero means and that

$$\langle C_x(z')C_x(z'')\rangle = \langle C_y(z')C_y(z'')\rangle = \Gamma(z' - z''), \quad (7a)$$

$$\langle C_x(z')C_y(z'')\rangle = 0. \quad (7b)$$

The case of arbitrary correlations Γ, or curvature spectra, could be handled as for the two-dimensional case.[5] However, for the sake of clarity, we shall consider mainly microscopic correlations,

$$\Gamma(z' - z'') = \gamma\delta(z' - z''), \quad (7c)$$

where γ represents the spectral power density of the processes and $\delta(\ )$ is the Dirac δ function.

Under those conditions, the probability $P^\dagger(E,\mu,z)$ that a ray be transmitted at z when the initial ray parameters are E and μ is found to obey the following (backward-Fokker-Planck) equation:

$$\gamma^{-1}\,\partial P^\dagger/\partial z = \partial P^\dagger/\partial E + (E - \overline{U})\partial^2 P^\dagger/\partial E^2 + \frac{1}{2}R\partial^2 P^\dagger/\partial\mu^2 + \mu\partial^2 P^\dagger/\partial E\partial\mu, \quad (8)$$

with the boundary condition that $P^\dagger(\Delta,\mu,z) = 0$, appropriate to the limit of small bending. It is interesting that the rate of increase of $\langle E\rangle$, given by the first term of Eq. (8), is independent of the index profile (for microscopic correlations) and equal to γ.

We have obtained Eq. (8) in two ways. The first was by deriving the Fokker–Planck equation for the probability expressed in terms of the phase–space variables $x,y,\dot{x},\dot{y}$,

$$\partial S/\partial z = (\gamma/2)(\partial^2 S/\partial\dot{x}^2 + \partial^2 S/\partial\dot{y}^2), \quad (9)$$

and going from the $\dot{x},\dot{y}$ variables to the E,μ variables, using Eq. (3), and holding x and y constant. Subsequently, averages over a ray period are performed. The second method consists of calculating directly the coefficients of the second-order terms in Eq. (8), using Eq. (4). We obtain readily, using Eq. (7c),

$$\gamma^{-1}l(E^2) = \overline{\dot{x}^2 + \dot{y}^2} = 2(E - \overline{U}), \quad (10a)$$

$$\gamma^{-1}l(E\mu) = \overline{x\dot{y} - y\dot{x}} = \mu, \quad (10b)$$

$$\gamma^{-1}l(\mu^2) = \overline{x^2 + y^2} = R, \quad (10c)$$

where we have introduced the notation,

$$l(ab) = \lim_{\Delta z \to 0} \langle [a(\Delta z) - a(0)][b(\Delta z) - b(0)]\rangle/\Delta z. \quad (10d)$$

Note that, even though we let Δz go to zero, it is understood that Δz remains much larger than a ray period. Bending is assumed to be so small that the ray parameters E,μ do not vary appreciably over a ray period; that is, we assume that γZ is small compared with unity. This condition (which, incidentally, is unrelated to the adjacent-mode-coupling approximation made in modal theories) by no means implies that the total microbending loss is small. Indeed, typical fibers involve millions of ray periods over their lengths. Thus, $\gamma Z \ll 1$ is consistent with $\gamma L \gg 1$.

The first-order term in Eq. (8) can be found by requiring that, when the probability is expressed in terms of the mode-numberlike variables I,μ, the equation becomes self-adjoint. This condition follows directly, for example, from a modal theory in the continuum limit if we assume that power coupling between modes is reciprocal. Specifically, setting

$$Q(I,\mu,z) = P^\dagger[E(I,\mu),\mu,z], \quad (11)$$

one can show after lengthy algebraic manipulations, using Eqs. (5d) and (6b), that the equation for Q is self-adjoint only if the coefficient of $\partial P^\dagger/\partial\mu$ in Eq. (8) is equal to zero and the coefficient of $\partial P^\dagger/\partial E$ is unity. The equation for $Q(I,\mu,z)$ is conveniently written in the form of a conservation equation,

$$(2/\gamma)\partial Q/\partial z + \partial J_I/\partial I + \partial J_\mu/\partial\mu = 0, \quad (12a)$$

where the I,μ components of the probability current $\mathbf{J}$ are, respectively,

$$J_I = -(\Phi K + ZI)\partial Q/\partial I + K\partial Q/\partial\mu, \quad (12b)$$

$$J_\mu = K\partial Q/\partial I - R\partial Q/\partial\mu, \quad (12c)$$

$$K \equiv \Phi R - Z\mu, \quad (12d)$$

and E,Z,Φ,R, defined earlier, are considered functions of I and μ. The terms in Eq. (12) can also be obtained directly from the expressions of $l(I^2)$, $l(I\mu)$, and $l(\mu^2)$ derived from Eq. (6c). It is not difficult to show that, for any function, $Q, J_I = 0$ along the $I = 0$ line (corresponding to helical rays, $K = 0$), and that $J_\mu = 0$ along the $\mu = 0$ line (corresponding to meridional rays, $\Phi = \pi$). These conditions hold also for arbitrary-curvature spectra. Thus, in order to solve Eq. (12), we have only to consider the boundary condition: $Q[(I(\Delta,\mu),\mu] = 0$. The factors in Eq. (12) are the power-coupling coefficients that modal theories should provide in the WKB approximation. This can be shown explicitly for meridional rays and arbitrary curvature spectra because the Fourier coefficients of $x(z)$ are simply related to the transition probabilities between modes.

The probability $P(E,\mu,z)\, dE\, d\mu$ that a ray has energy between E and $E + dE$ and angular momentum between μ and $\mu + d\mu$ at z obeys an equation (the so-called forward-Fokker–Planck equation) that is the adjoint of Eq. (8). One can easily show that $P = ZP^\dagger$, using again Eqs. (5d) and (6b).

Power-Law Profiles

For power-law profiles,[7]

$$U(r) = \Delta(r/r_c)^{2\kappa}, \quad (13)$$

where Δ denotes the relative index change, r_c the core radius, and $0 < \kappa < \infty$ a parameter, the virial theorem shows that

$$E - \overline{U} = [\kappa/(1 + \kappa)]E. \quad (14)$$

If we consider solutions of Eq. (8) of the form

$$P^\dagger(E,\mu,z) = P_m{}^\dagger(E)\exp(-\lambda_m z); P_m{}^\dagger(\Delta) = 0, \quad (15)$$

where λ_m represents the microbending loss of the statistical mode of order m, closed-form solutions are found in terms of Bessel functions of the order $\nu = 1/\kappa$. We shall give here only the expression of the microbending loss of the fundamental statistical mode,

$$\lambda = u_\nu{}^2(\gamma/8\Delta); J_\nu(u_\nu) = 0. \quad (16)$$

For square-law profiles, $\kappa = 1$, for example, the loss is

$$\text{Loss} = 7.969\ \gamma/\Delta \text{ dB/unit length}, \quad (17)$$

as found earlier by different methods.[8] Note that although $P^\dagger$, as considered above, is independent of μ, the power Q in mode I,μ does depend on μ, according to Eq. (11), because E depends on μ. Let us also observe that, contrary to what is implied in Refs. 1 and 7, the modes of propagation in fibers with power-law profiles are not degenerate, except for $\kappa = 1$; that is, E is not a simple combination of the two mode numbers. [See, for example, Keller and Rubinow's asymptotic expression for step-index fibers, quoted in Eq. (5.104) of Ref. 6.] Thus, it is not permissible to use a single compound mode number, except perhaps as an approximation for near-square-law profiles.

For nonuniform curvature spectra, the factors in Eq. (8) need to be expressed as infinite sums. The factor γR of $\partial^2 P^\dagger/\partial\mu^2$, for example, should be replaced by a term of the form

$$\sum_{n=1}^{\infty} f_n f_n{}^* g[(2n\pi - \Phi)/Z], \quad (18)$$

where the f_n denote the Fourier coefficients of the complex periodic function,

$$[x(z) + iy(z)]\exp(-iz\Phi/Z), \quad (19)$$

and $x(z)$, $y(z)$ denote, as before, a ray trajectory in the circularly symmetric fiber. The function $g(a)$ denotes the spectral power density of the curvature process at the spatial frequency a. For the case of microscopic correlations considered earlier, g is the constant γ. The adjacent-mode-coupling approximation amounts to keeping only the first term in the series in Eq. (18). For uniform spectra and step-index fibers, this involves an error of the order of 20%. The error may be much larger if the spectral density increases in the relevant range of spatial frequencies, as is the case, in particular, for near-sinusoidal deformations. For helical rays, only the first term in the series, Eq. (18), is different from zero, since helical rays project a sinusoids on both the x–z and y–z planes.

Finally, let us observe that, contrary to what has been done in Ref. 1, we did not find it necessary to interpolate between the step-index and the square-law values to obtain the diffusion coefficients. Admittedly, not all of our results are in closed form, but perhaps one cannot go much further analytically without approximations. The statistical modes discussed in this Letter are essential to obtaining the impulse response of a fiber, whether one uses Olshansky's perturbation technique[1] or integrates time along the ray trajectories.[9]

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