

RAY THEORY OF MICROBENDING

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A ray theory is given for randomly bent (two-dimensional) optical fibers that have arbitrary index profiles and arbitrary curvature spectra. Simple closed form results are given for power-law profiles and spectra. No approximation is made besides the small bending approximation and the paraxial ray optics approximation. In particular, the coupling between all modes is effectively taken into account.

1. Introduction

The random distortion of the axis of optical fibers has profound effects on their transmission characteristics. We investigate these effects with the help of a ray theory, which is sufficiently accurate when the fiber carries a large number of modes, typically more than 10. For the sake of clarity, we restrict ourselves to two-dimensional fibers.

2. General theory

It is well known [1, 2] that the transmission of rays in randomly curved fibers is analogous to the motion of mechanical oscillators driven by random forces. Thus, if z denotes the length along the curved fiber axis, and $n(x)$ the index profile, paraxial ray trajectories $x = x(z)$ obey the equation

$$\ddot{x} = F(x) + C(z), \quad (1)$$

where upper dots denote differentiation with respect to z , and we have defined

$$F(x) = -dU(x)/dx, \quad U(x) = 1 - n(x)/n(0). \quad (2a, b)$$

The fiber axis curvature $C(z)$ is random.

The purpose of this paper is to evaluate the probability $P(\dot{x}, x) d\dot{x} dx$ that a ray has slope comprised between \dot{x} and $\dot{x} + d\dot{x}$, and position comprised between x and $x + dx$, at some z , for specified initial conditions. The steady state microbending losses are directly related to the equation for P .

3. Straight fibers

A few results applicable to straight fibers are recalled here. The amplitude of a ray trajectory $x = x(z)$ can be characterized by the parameter

$$E = U(x) + \dot{x}^2/2, \quad (3)$$

which is analogous to the mechanical energy. It can also be characterized by the area enclosed in phase space by the ray trajectory,

$$I = \iint d\dot{x} dx \quad (4)$$

which is analogous to the mechanical action. In the language of wave optics, I is essentially the mode number and E the propagation constant. Alternative useful expressions for I are

$$I = \oint \sqrt{2[E - U(x)]} dx = Z\bar{x}^2, \quad (5)$$

where Z denotes the ray period and overbars denote averages over a ray period. They follow from earlier definitions.

If we differentiate eq. (5) under the sum sign and use eq. (3), we obtain

$$Z = dI/dE, \quad (6)$$

where I is considered a function of E .

Power-law profiles [3] are of particular interest. We set

$$n(x) = n_0 [1 - \Delta(x/x_c)^{2\kappa}], \quad (7a)$$

$$U(x) = \Delta(x/x_c)^{2\kappa}. \quad (7b)$$

For that case we have [4]

$$I(E, \kappa) = x_c \sqrt{2\Delta} (E/\Delta)^{(\kappa+1)/2\kappa} g(\kappa), \quad (8)$$

where

$$g(\kappa) = 2\sqrt{\pi}\Gamma(1 + 1/2\kappa)/\Gamma(3/2 + 1/2\kappa) \quad (9)$$

and $\Gamma(\cdot)$ denotes the gamma function. We have, in particular, $g(1) = \pi$, $g(\infty) = 4$. The expression for the ray period follows, as we have seen, by differentiation

$$Z(E, \kappa) = (x_c/\sqrt{2\Delta}) [(\kappa + 1)/2\kappa] (E/\Delta)^{\nu} g(\kappa), \quad (11)$$

$$\nu = (1 - \kappa)/2\kappa.$$

In particular

$$\left. \begin{aligned} I(E, 1) &= \pi x_c \sqrt{2\Delta} (E/\Delta) \\ Z(E, 1) &= 2\pi x_c / \sqrt{2\Delta} \end{aligned} \right\} \text{square-law} \quad (12)$$

$$\left. \begin{aligned} I(E, \infty) &= 4x_c \sqrt{2\Delta} (E/\Delta)^{1/2} \\ Z(E, \infty) &= (4x_c/\sqrt{2\Delta}) (E/\Delta)^{-1/2} \end{aligned} \right\} \text{step index} \quad (13)$$

4. The Fokker-Planck equation

Let us go back now to the general equation (1), and assume that the fiber axis curvature $C(z)$ has zero means and microscopic correlation

$$\langle C(z) C(z') \rangle = \gamma \delta(z - z'). \quad (14)$$

In eq. (14), $\delta(\cdot)$ denotes Dirac's δ -function and γ the

power spectral density of the process. Under those conditions, it is well known (see ref. [5], eq. (4-254), in the limit of negligible damping) that the probability $P(E, z) dE$ that a ray has energy between E and $E + dE$ at z obeys the equation

$$\partial P/\partial z = -\partial(AP)/\partial E + (1/2) \partial^2 [B(E)P]/\partial E^2, \quad (15a)$$

where

$$A = \gamma/2, \quad B(E) = \gamma I/(dI/dE) = \gamma I/Z = \gamma \bar{x}^2. \quad (15b, c)$$

I is the ray action defined earlier in eq. (5).

For comparison with modal theory, it is useful to write eq. (15) in the self-adjoint form

$$\partial Q/\partial z = \partial [D(I) \partial Q/\partial I]/\partial I, \quad (16)$$

where we have used I instead of E to characterize the ray amplitude, and where we have introduced a normalized probability density $Q(I, z)$, related to P by

$$Q(I, z) dI = P(E, z) dE. \quad (17)$$

The diffusion coefficient $D(I)$ is equal to $BZ^2/2$.

In the special case of power-law profiles, eq. (15a) becomes, using eq. (8)

$$\frac{\partial P}{\partial z} = -\frac{\partial}{\partial E} \left(\frac{\gamma}{2} P \right) + \frac{1}{2} \frac{\partial^2}{\partial E^2} \left(\frac{2\kappa}{\kappa+1} \gamma E P \right). \quad (18)$$

It can be shown that the ratio $P(\Delta, z)/P(0, z)$ tends to zero as $\gamma^{2/5}$ as $\gamma \rightarrow 0$. Thus, in the limit of small bending, we have the boundary condition $P(\Delta, z) = 0$. The closed-form stationary solutions of eq. (18), called statistical modes, are obtained by setting $P(E, z) = P_m(E) \exp(-\lambda_m z)$. We have the unnormalized densities

$$P_m(E) = (E/\Delta)^{\nu/2} J_{\nu}(u_{\nu m} \sqrt{E/\Delta}), \quad (19)$$

where

$$\nu = (1 - \kappa)/2\kappa \quad (20a)$$

and

$$\lambda_m = [\kappa/(1 + \kappa)] u_{\nu m}^2 (\gamma/4\Delta). \quad (20b)$$

$J_{\nu}(\cdot)$ denotes the Bessel function of order ν and $u_{\nu m}$ the m th zero of J_{ν} , $m = 0, 1, \dots$. The lowest steady-state microbending loss is obtained for $m = 0$. The loss expressed in dB/unit length is

$$\alpha_m = 4.34 \lambda_m. \quad (21)$$

The left-hand-side of eq. (21) is plotted as a function

of κ in fig. 1. We note that for white curvature spectra the microbending loss is not very sensitive to the profile, at least from $\kappa = 1$ to $\kappa = \infty$, as noted earlier [2].

The probability density in eq. (19) is everywhere positive for $m = 0$, but, for $m > 0$, it may be negative for some values of E . Such statistical modes do have an independent physical meaning, because one may always add to them a constant large enough to make them everywhere positive. Physically, this constant represents radiation or cladding power.

The irradiance is more readily measured than the probability density. It is obtained by integrating P over \dot{x} while x , and therefore U , is maintained a constant. We obtain

$$S(x) = \int_{x_c}^1 \gamma^{\nu} J_{\nu}(u_{\nu m} \gamma^{\nu}) dy, \quad (22)$$

where

$$\chi = x/x_c.$$

This integral, unfortunately, can be evaluated only numerically.

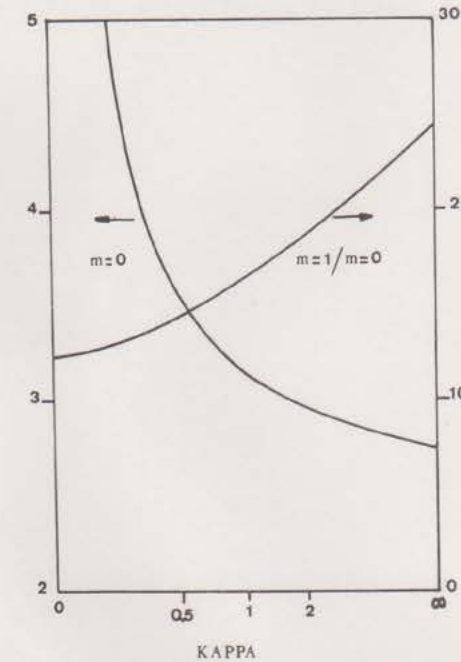


Fig. 1. Curve $m = 0$: variation of the normalized steady state microbending loss: $(\Delta/\gamma) \times$ loss in dB/unit length as a function of the profile parameter κ (note: $\kappa = 1$, square-law profile). Curve $m = 1/m = 0$: loss ratio of the first two statistical modes. The horizontal scale is linear in $\kappa/(\kappa + 1)$.

5. Non-uniform curvature spectra

The spectral density of the fiber axis curvature is usually not a constant, as we assumed earlier. However, the general form in eq. (16) is the same for non-white spectra [6]. Thus, we only have to calculate, according to the standard Fokker-Planck theory

$$B(E) = \lim_{\Delta z \rightarrow 0} \langle [E(\Delta z) - E(0)]^2 \rangle / \Delta z, \quad (23a)$$

where

$$E(\Delta z) - E(0) = \int_0^{\Delta z} \dot{x} C(z) dz. \quad (23b)$$

Thus

$$\begin{aligned} \langle [E(\Delta z) - E(0)]^2 \rangle &= 2 \int_0^{\Delta z} \Gamma(\xi) d\xi \int_0^{\Delta z - \xi} \dot{x}(z) \dot{x}(z + \xi) dz, \end{aligned} \quad (24a)$$

where

$$\Gamma(\xi) = \langle C(z) C(z + \xi) \rangle, \quad \Gamma(\Delta z) \approx 0, \quad (24b)$$

$$\begin{aligned} \int_0^{\Delta z - \xi} \dot{x}(z) \dot{x}(z + \xi) dz &\approx \frac{1}{2} \Delta z \sum_{1,3} \dot{x}_n^2 \cos(2\pi n \xi / Z) \end{aligned} \quad (25)$$

if we set

$$\begin{aligned} \dot{x}(z) &= \sum_{1,3} \dot{x}_n \cos(2\pi n z / Z) \\ \dot{x}_n &= (8/Z) \int_0^{Z/4} \dot{x}(z) \cos(2\pi n z / Z) dz \end{aligned} \quad (26)$$

and assume that $x(0) = 0$. Thus

$$B = \frac{1}{2} \sum_{1,3} \dot{x}_n^2 G(n/Z), \quad (27)$$

where

$$G(k) = \int_{-\infty}^{+\infty} \Gamma(\xi) \cos(2\pi k \xi) d\xi \quad (28)$$

denotes the power spectral density of the $C(z)$ process. If we are dealing with power-law profiles, and we assume that G is proportional to k^q , B in eq. (27) is found to be proportional to $E^{1-q\nu}$, provided the sum converges. For step-index fibers

$$\dot{x}_n^2 = (4/\pi)^2 2E/n^2, \quad n = 1, 3 \dots \quad (30)$$

Neglect of the coupling between non-adjacent modes introduces for that case an error of the order of 20%. Note further that in the theory in ref. [7], the mode coupling coefficients for arbitrary κ are merely interpolated between the square-law and step-index values. None of these approximations has to be made here.

The results that we have presented are consistent with previously published results for special cases [8–10]. They are applicable to arbitrary smooth profiles and arbitrary regular curvature spectra. For profiles other power-law profiles, simple numerical integrations are needed. The direct tracing of rays through randomly bent fibers is very costly in term of computer time.

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Note that the numerical factors in eqs. (32) and (33) of this paper should read 0.00431 and 0.298 respectively; eq. (34) should read: total loss/(transm. cap. impr.)² = 0.128 dB.

