Rate-equation approach to atomic-laser light statistics

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We consider three- and four-level atomic lasers that are either incoherently (unidirectionally) or coherently (bidirectionally) pumped, the single-mode cavity being resonant with the laser transition. The intracavity Fano factor and the photocurrent spectral density are evaluated on the basis of rate equations. According to that approach, fluctuations are caused by jumps in active *and* detecting atoms. The algebra is simple. Whenever a comparison is made, the expressions obtained coincide with the previous results. The conditions under which the output light exhibits sub-Poissonian statistics are considered in detail. Analytical results, based on linearization, are verified by comparison with Monte Carlo simulations. An essentially exhaustive investigation of sub-Poissonian light generation by three- and four-level lasers has been performed. Only special forms were reported earlier.

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I. INTRODUCTION

Interest in the statistics of the light emitted by atomic lasers has been recently revived as a result of the fabrication of microlasers [1]. The purpose of the present paper is to emphasize that the rate-equation approach is a conceptually and algebraically simple method, even when the generated light exhibits sub-Poissonian statistics. Additionally, we derive analytical expressions applicable to realistic lasers.

An introduction to the present rate-equation method as it pertains to sub-Poissonian light generation can be found in tutorial papers [2,3]. The second of these two papers considers as a starting point an isolated single-mode cavity containing N two-level atoms. From the equal-weight rule of statistical mechanics, it is concluded that the Fano factor (ratio of the variance of the number m of photons in the cavity to its average value) is equal to 1/2 at equilibrium if, initially, all the atoms are in their upper states and the cavity is empty [4]. This equilibrium situation is easily generalized to the case of N four-level atoms with levels $|0\rangle, |1\rangle, |2\rangle, |3\rangle$, with the $|0\rangle - |1\rangle$ and $|2\rangle - |3\rangle$ transitions coupled to $M \gg N \gg 1$ modes. In that case, the Fano factor relating to the single-mode cavity field essentially vanishes. The case of isolated cavities in a state of equilibrium does not tell, of course, the whole story as far as lasers are concerned. But the discussion given by Arnaud [3] shows how the pump-driven systems may be treated through a natural (though somewhat heuristic) manner. Only one-photon processes are considered in that earlier paper, as well as in the present one.

The rate equations treat the number of photons in the cavity as well as the numbers of atoms in each state as classical random functions of time. The light field is quantized as a result of matter quantization and conservation of energy, but not directly. The rate equations should be distinguished from the semiclassical theories in which the optical field is driven by the atomic dipole expectation values. The theory employed in this paper rests instead on the consideration of transition probabilities, as in the Loudon [5] treatment of optical amplifier noise, for example. Let us emphasize that every absorption event reacts on the number of light quanta in the optical cavity, particularly those occurring in light detectors, irrespective of their locations. A single ideal detector is considered in the present treatment. It collects all the generated light and has unity quantum efficiency. The semiclassical theories are unable to explain sub-Poissonian light statistics because the light generation process and the light detection process are considered separately.

Analytical expressions are obtained from the rate equations in a straightforward manner as solutions of a few linear equations. The great advantage of our approach is that it is easily applied to realistic situations in which many atomic levels are involved. The analytical expressions are sometimes too lengthy to be exhibited in a paper. But symbolic calculus enables us to easily determine the optimum conditions of operation, for example the parameter values that minimize the photodetection noise at some prescribed Fourier frequency.

Our results always agree with more rigorous treatments when the number of atoms is large compared with unity, and

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all coherences (off-diagonal density-matrix elements) are damped at rates faster than the populations. The active medium is supposed to be strongly homogeneously broadened so that atomic polarizations may be adiabatically eliminated [6]. It is also supposed that transitions other than those relating to the atom-cavity interaction are incoherent. The rate of spontaneous decay from the upper to the lower working levels may take any value, and in particular be set equal to zero. The latter situation ($\gamma=0$ in our notation) occurs in ideal "cavity QED lasers" [7,8].

Let us now briefly review previous treatments of Poissonian and sub-Poissonian light generation. One of the best-known laser-noise theory is probably that of Scully and Lamb [9]. Incoherent pumping is modeled by the independent injection of two-level atoms in the optical cavity. This model leads to a photocount statistics which is, at best, Poissonian. More recently, Khazanov *et al.* [10], Ralph and Savage [11], and Ritsch *et al.* [12] considered the situation in which the populations of pumping levels may fluctuate. At first, it would seem that this may only increase the noise. It turns out, however, that the population fluctuations are correlated in such a way that the output light fluctuations may be sub-Poissonian. It is difficult to pin point a simple intuitive explanation. It has been observed, however, that when the lasers are pumped through a cascade of intermediate levels, pumping tends to be regular [13,14], a situation somewhat similar to laser-diodes high-resistance driving conditions. Other means of generating sub-Poissonian light have been considered. Golubev and Sokolov [15] were the first in 1984 to point out that lasers with nonfluctuating pumps should emit sub-Poissonian light. This conclusion has been verified experimentally by Machida *et al.* [16] with the help of laser diodes driven by high-impedance electrical sources. The Scully and Lamb model has been generalized to account for the regular atom injection [17,18]. Kolobov *et al.* [19] made the interesting observation that the photodetection rate spectral density may be below the shot-noise level at *nonzero* Fourier frequencies in the case of *Poissonian* pumps. However, the photodetection rate spectral density remains at the shot-noise level at *zero* frequency. Accordingly, such lasers do not generate sub-Poissonian light in the sense defined earlier. It has been shown that three-level lasers, with coherent decay to the ground state [20,21], and Raman lasers may generate sub-Poissonian light [22]. We will not consider here these more exotic configurations. A review is in Ref. [23]. Many other relevant references may be found in Ref. [7].

V-type incoherently pumped lasers were treated earlier by Khazanov *et al.* [10]. Ralph and Savage [11,24] extended the analysis to incoherently pumped Λ-type lasers and four-level lasers. Ritsch *et al.* [12] gave a description of four-level lasers for the two pumping schemes. These previous results are exactly recovered from the present rate-equations method. In particular, the expression for the internal cavity statistics of four-level atoms with negligible spontaneous decay between working levels, previously given in Eq. (4) of Ref. [12] is recovered exactly [see Eq. (17) of the present paper]. If the upper and lower decay times of four-level atoms tend to zero, the laser is equivalent to a two-level atomic laser with Poissonian pump [25]. In that limit, the expressions reported

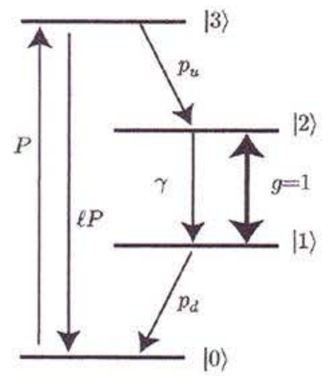

FIG. 1. Level schemes for a four-level atomic laser. For incoherent pumping, $\ell=0$. For coherent pumping, $\ell=1$. Assuming a unitary gain g subsequently normalizes all rates to the corresponding time unit.

in Arnaud and Estèban [26] in 1990 are recovered. The formulas derived in the present paper for coherently pumped three-level lasers appear to be new as well as results considering arbitrary decay rates and arbitrary cavity losses, to best of our knowledge.

We first give in Sec. II details on our rate-equation approach to laser modeling. The weak-noise approximation is discussed and compared to the results of Monte Carlo simulations. When both the number of atoms and the pumping level increase, the Monte Carlo simulation computing time becomes prohibitively large. In that case, analytical results are essential, see Sec. III. The photocurrent spectral density at zero frequency, the photocurrent spectrum, and the intracavity Fano factor are obtained and illustrated for various parameter values.

II. LASER MODEL

The active medium is a collection of N identical four-level atoms, see Fig. 1. Level separations are supposed to be large compared with k_BT, where T denotes the optical cavity temperature and k_B is the Boltzmann's constant, so that thermally induced transitions are negligible. Levels $|1\rangle$ and $|2\rangle$ are resonant with the field of a single-mode optical cavity.

The probability per unit time that an electronic transition from level $|1\rangle$ to level $|2\rangle$ occurs is taken as equal to m, and the probability of an electronic transition from $|2\rangle$ to $|1\rangle$ as $m+1$ (the qualification "per unit time" is henceforth omitted for the sake of brevity). This amounts to selecting a time unit whose typical value depends on the gain medium. As a secondary result, each laser transition rate and laser parameter is then normalized to this time unit. Decay from level $|2\rangle$ to level $|1\rangle$ is allowed with probability γ. This decay may be either nonradiative or involve spontaneous emission into electromagnetic modes besides the one of interest. The photons are absorbed with probability αm, where α denotes a constant, the absorbing atoms residing most of the time in their ground state. These absorbing atoms model the transmission of light through mirrors with subsequent absorption by a detector.

"Incoherent" pumping promotes electrons from level $|0\rangle$

to level $|3\rangle$ with probability P. When transitions from $|0\rangle$ to $|3\rangle$ and from $|3\rangle$ to $|0\rangle$ are both allowed with equal probabilities, the pumping process is called "coherent," following an accepted terminology. We find it convenient to denote by ℓP the $|3\rangle\rightarrow|0\rangle$ transition probability, with $\ell=0$ for incoherent pumping and $\ell=1$ for coherent pumping. "Coherent" pumping is physically realized by submitting the atoms to strong optical fields nearly resonant with the $|0\rangle\rightarrow|3\rangle$ transition. This pumping field may possibly originate from the frequency-filtered thermal radiation. Levels $|0\rangle$ and $|3\rangle$ need not be sharp. Instead, they may consist of narrow bands for improved coupling to the broadband pumps. One-way incoherent pumping would be appropriate to describe laser-diode pumps. Laser diodes have been treated previously on the basis of the rate equations in Ref. [3]. Spontaneous decay from level $|3\rangle$ to the upper working level $|2\rangle$ occurs with probability p_u, and spontaneous decay from the lower working level $|1\rangle$ to the ground level with probability p_d.

Since thermally induced transitions have been neglected, the three-level V-type scheme (obtained when levels $|0\rangle$ and $|1\rangle$ collapse) and the three-level Λ-type scheme (obtained when levels $|3\rangle$ and $|2\rangle$ collapse) are *not* special cases of the four-level scheme when "coherent" pumping is considered. They need to be treated separately using the same general approach as discussed below for the four-level laser. The detailed formulas, however, will be omitted for the sake of brevity.

According to the previous model, the laser-detector assembly is treated as a birth-death Markov process equivalent to the master equation used by Rice and Carmichael [7]. It is thus suitable for Monte Carlo simulations [27,28], thereby illustrating the evolution of the number m of photons in the cavity from which the Fano factor $\mathscr{F}=\mathrm{var}(m)/\langle m\rangle$ can be extracted. Similarly, the instants t_k when photons are being absorbed provide us with the spectral density of the photocurrent, whose normalized value $\mathscr{S}$ is unity for Poisson processes. In the following, the normalized spectrum is denoted by $\mathscr{S}(\Omega)$, where Ω, the normalized Fourier angular frequency, is called for short "frequency."

Monte Carlo calculations have been applied to an incoherently pumped V-type three-level atomic laser. Our purpose here is to exemplify the results that can be derived from the rate-equation method, and to provide a check on the validity of the linearization procedure to be later employed, see Sec. III. The calculated intracavity Fano factor is represented in Fig. 2(a). Spiking at threshold as well as sub-Poissonian light statistics at high pumping are obtained in good agreement with the data of Koganov and Shuker [29]. The normalized spectral density $\mathscr{S}(\Omega)$ is represented in Fig. 2(b) for two sets of parameter values. $\mathscr{S}(\Omega)$ is first evaluated for each Monte Carlo run [30] and refined using a smoother power spectral density estimator [31,32]. Averaging over runs and concatenating neighboring frequencies produce the final data with error bars. Again the low-frequency light statistics is found to be sub-Poissonian for appropriate pumping levels.

There is a fair agreement between the Monte Carlo simulations and the analytical formulas to be subsequently re-

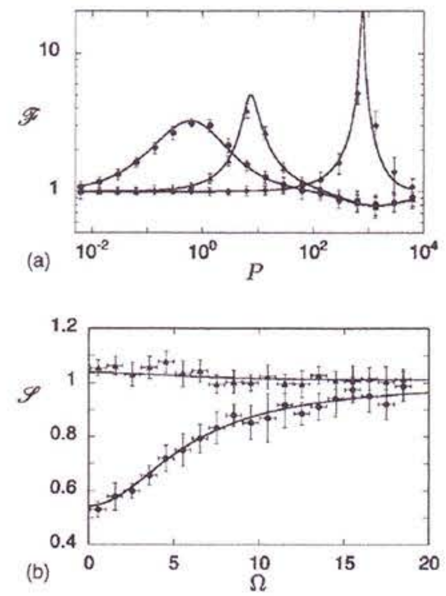

FIG. 2. Monte Carlo calculation results of an incoherently pumped V-type laser. The parameters are $N=100$ atoms, $p_u=632$, $\alpha=6.32$. ● shows $\gamma=0$; ▲ shows $\gamma=6.32$; ◆ shows $\gamma=632$. Error bars are for a 95% confidence level. (a) Intracavity Fano factor $\mathscr{F}$ as a function of the normalized pumping rate P. (b) Normalized photocurrent spectral density $\mathscr{S}$ as a function of the normalized Fourier frequency Ω. Plain lines are analytical (see text).

ported. We conclude from Fig. 2 that the linearization procedure is a valid one, at least when the number of active atoms exceeds about one hundred.

Finally, notice that even with one billion photon-absorption events, Monte Carlo spectra of Fig. 2(b) exhibit large error bars. The analytical method is thus to be preferred when available.

III. ATOMIC LASER LIGHT STATISTICS

Let n_j, $j=0,1,2,3$, denote the number of atoms in state j, with

$$n_0+n_1+n_2+n_3=N. \tag{1}$$

Let $\mathcal{J}$ denote the net pumping rate, $\mathcal{R}$ the net stimulated rate, $\mathcal{U}$ and $\mathcal{D}$ the upper and lower decay rates, $\mathcal{S}$ the spontaneous decay rate from the upper to the lower working levels, and $\mathcal{Q}$ the photon-absorption rate. The steady-state conditions then read

$$\mathcal{J}=\mathcal{U}=\mathcal{D}=\mathcal{R}+\mathcal{S}, \qquad \mathcal{Q}=\mathcal{R}, \tag{2}$$

where according to the probabilities enumerated in Sec. II, eight kinds of events may occur in the course of time,

$$\mathcal{J}=Pn_0-\ell Pn_3, \qquad \mathcal{U}=p_u n_3, \tag{3a}$$

$$\mathcal{R}=(m+1)n_2-mn_1, \qquad \mathcal{D}=p_d n_1, \tag{3b}$$

$$\mathcal{S}=\gamma n_2, \qquad \mathcal{Q}=\alpha m. \tag{3c}$$

Equations (1), (2), and (3) provide the steady-state atomic populations n_i and photon number m. In particular,

$$m=\frac{1}{2}(\mathcal{B}+\sqrt{\mathcal{B}^2+4\mathcal{P}\mathcal{N}}), \tag{4}$$

where

$$\mathcal{N}=\frac{N}{\alpha}, \tag{5a}$$

$$\mathcal{P}=\left[\frac{1}{P}+\frac{2}{p_d}+\frac{1+\ell}{p_u}\right]^{-1}, \tag{5b}$$

$$\mathcal{B}=\mathcal{P}\left[\mathcal{N}-1+\frac{1}{p_d}(1+\gamma-\mathcal{N}\gamma)\right]-\gamma-1. \tag{5c}$$

For moderate pump powers above threshold, m increases linearly with P. The intercept with the $m=0$ axis may be used to define the threshold pump power.

Our analytical results rest on a weak-noise approximation. Populations split into steady-state values and fluctuations. For example, the instantaneous photon number m is written as $\langle m\rangle+\Delta m$, where $\langle m\rangle$ denotes the steady-state value. The rates split into steady-state values and fluctuations consisting of a deterministic function of the population fluctuations and Langevin "forces." These forces are δ-correlated in time (white noise) and uncorrelated with one another. For example, $\mathcal{J}$ splits into $J=\langle\mathcal{J}\rangle$ and ΔJ. The latter is the sum of a deterministic function of the population fluctuations, and a Langevin force $j(t)$ expressing the jump process randomness. Thus

$$\mathcal{J}\equiv J+\Delta J, \qquad \mathcal{U}\equiv U+\Delta U, \tag{6a}$$

$$\mathcal{R}\equiv R+\Delta R, \qquad \mathcal{D}\equiv D+\Delta D, \tag{6b}$$

$$\mathcal{S}\equiv S+\Delta S, \qquad \mathcal{Q}\equiv Q+\Delta Q, \tag{6c}$$

where

$$\Delta J=P\Delta n_0-\ell P\Delta n_3+j, \qquad \Delta U=p_u\Delta n_3+u, \tag{7a}$$

$$\Delta R=(m+1)\Delta n_2-m\Delta n_1+(n_2-n_1)\Delta m+r,$$

$$\Delta D=p_d\Delta n_1+d, \tag{7b}$$

$$\Delta S=\gamma\Delta n_2+s, \qquad \Delta Q=\alpha\Delta m+q. \tag{7c}$$

A first-order variation of the expressions in Eq. (3) has been performed.

Conservation of the rates gives

$$\Delta J=\Delta U=\Delta D=\Delta R+\Delta S, \qquad \Delta Q=\Delta R. \tag{8}$$

Since the total number N of atoms is constant, we have

$$\Delta n_0+\Delta n_1+\Delta n_2+\Delta n_3=0. \tag{9}$$

Replacing atomic populations and photon numbers by their steady-state values, the above set of equations can be solved. In particular, ΔQ is a linear combination of the Langevin forces,

$$\Delta Q=\sum_{z\in\{j,d,u,q,r,s\}} c_z z, \tag{10}$$

where the c_z are real coefficients that depend on the parameters N, P, ℓ, p_u, p_d, γ, and α. The normalized zero-frequency photocurrent spectral density is of the form

$$\mathcal{S}=\frac{1}{\alpha m}\sum_{z\in\{j,d,u,q,r,s\}} c_z^2\sigma_z, \tag{11}$$

where σ_z denotes the spectral density value of the Langevin noise source z, equal to average rates;

$$\sigma_j=Pn_0+\ell Pn_3, \qquad \sigma_u=p_u n_3, \tag{12a}$$

$$\sigma_r=(m+1)n_2+mn_1, \qquad \sigma_d=p_d n_1, \tag{12b}$$

$$\sigma_s=\gamma n_2, \qquad \sigma_q=\alpha m. \tag{12c}$$

When these expressions are introduced in Eq. (11) an analytical expression of $\mathcal{S}$ is obtained. Three special cases are considered below (a) $\gamma=0$ and m large compared with unity, (b) $N\gg\alpha$, and (c) $\gamma=0$ and $N\gg\alpha$.

(a) If spontaneous decay is negligible ($\gamma=0$), Eq. (11) yields

$$\mathcal{S}=1+\frac{2}{(\mathcal{N}-1)^2}+\frac{8\mathcal{P}^2}{p_d^2}+\frac{(6-4\mathcal{N})\mathcal{P}}{(\mathcal{N}-1)p_d}+\frac{2(1+\ell)\mathcal{P}^2}{p_u^2}+\frac{2\mathcal{P}(2\mathcal{P}-p_d)}{p_d p_u}, \tag{13}$$

where $\mathcal{P}$ and $\mathcal{N}$ are defined in Eq. (5). The normalized spectral density is unity at low and high pumping levels. For some constant $\mathcal{N}$ value, $\mathcal{S}$ reaches its minimum value

$$\mathcal{S}_{min}=\frac{2\mathcal{N}(\mathcal{N}-1)+11+\ell(4\mathcal{N}(\mathcal{N}-1)+15)}{2(3+4\ell)(\mathcal{N}-1)^2}, \tag{14}$$

when

$$\frac{P}{p_u}=\frac{1}{1+\ell}, \qquad \frac{P}{p_d}=\frac{\mathcal{N}(1+2\ell)-(2+3\ell)}{(1+\ell)(2\mathcal{N}-1)}. \tag{15}$$

(b) When $N\gg\alpha$, Eq. (11) yields

$$\mathcal{S}=1+\frac{2\gamma}{p_d-\gamma}-\frac{4\mathcal{P}+2\gamma}{p_d}+\frac{8\mathcal{P}(\mathcal{P}+\gamma)}{p_d^2}-\frac{8\mathcal{P}^2\gamma}{p_d^3}+\frac{2\mathcal{P}(\gamma-p_d)(p_d-2\mathcal{P})}{p_d^2 p_u}-\frac{2(1+\ell)\mathcal{P}^2(\gamma-p_d)}{p_d p_u^2}. \tag{16}$$

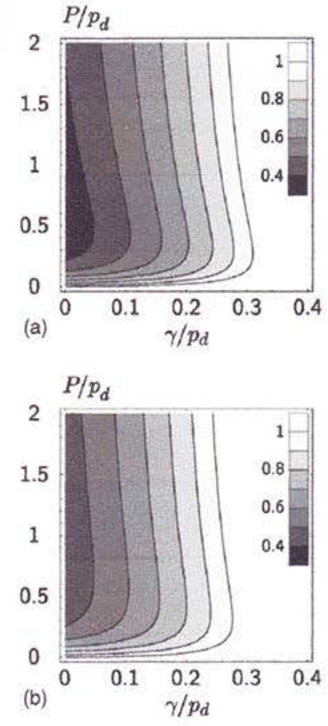

FIG. 3. Contour plots of the zero-frequency normalized photocurrent spectral density $\mathcal{S}$ for four-level atomic lasers as a function of γ/p_d and P/p_d. Parameters are $N=10^5$ atoms, $\alpha=6.32$, $p_d=632$. (a) Incoherent pumping, $p_u=316$. (b) Coherent pumping, $p_u=949$. In the white area, the light statistics is super-Poissonian.

Figure 3 gives the normalized zero-frequency photocurrent spectral density $\mathcal{S}(P/p_d,\gamma/p_d)$ in the form of contour plots, selecting $P/p_u=2P/p_d$ for the case of incoherent pumping and $P/p_u=\frac{2}{3}P/p_d$ for the case of coherent pumping. The darker the area, the lower is the spectral density. Since dark areas are wider in Fig. 3(a) than in Fig. 3(b), incoherent pumping is to be preferred. Figure 3 shows that sub-Poissonian light generation by optically pumped four-level atomic lasers is robust against moderate spontaneous decay and pumping level changes. The optimum conditions (darkest areas) are defined in Eq. (15). But small departures from these conditions do not increase the noise much.

(c) If both $\gamma=0$ and $N\gg\alpha$, the spectral density obtained either by setting $\mathcal{N}=\infty$ in Eq. (13) or $\gamma=0$ in Eq. (16), reads

$$\mathcal{S}=1-\frac{2Pp_d p_u[p_d+2(P+P\ell+p_u)]}{[2Pp_u+p_d(P+P\ell+p_u)]^2}, \tag{17}$$

an expression that coincides with Eq. (4) of Ref. [12]. The absolute minimum value and corresponding conditions are obtained from Eqs. (14) and (15),

$$\mathcal{S}_{min}=\frac{1+2\ell}{3+4\ell}, \qquad \frac{P}{p_u}=\frac{1}{1+\ell}, \qquad \frac{P}{p_d}=\frac{1+2\ell}{2(1+\ell)}. \tag{18}$$

TABLE I. Minimum value of the zero-frequency photocurrent spectral density $\mathcal{S}_{min}$ and intracavity Fano factor $\mathcal{F}$ for three- and four-level atomic lasers. The conditions on P, p_u, and p_d are given. Spontaneous decay from the upper working level is neglected and $N\gg\alpha$ is assumed.

Laser	$\mathcal{S}_{min}$	$\mathcal{F}$	Conditions
Λ-type three-level [a]	1/2	3/4	$p_d=2P$
Λ-type three-level [b]	2/3	5/6	$p_d=3P$
V-type three-level [a]	1/2	3/4	$p_u=\frac{1}{2}P$
V-type three-level [b]	5/6	11/12	$p_u=\frac{3}{2}P$
Four-level [a]	1/3	2/3	$p_u=P$, $p_d=2P$
Four-level [b]	3/7	5/7	$p_u=2P$, $p_d=\frac{4}{3}P$

[a]Incoherent pumping.
[b]Coherent pumping.

For incoherent pumping, $\ell=0$, we have therefore $\mathcal{S}_{min}=1/3$ when $p_u=P$ and $p_d=2P$. For coherent pumping, $\ell=1$, we have $\mathcal{S}_{min}=3/7$ when $p_u=2P$ and $p_d=4/3P$.

Table I gives the minimum spectral density values achievable with optically pumped three- and four-level atomic lasers. Under the conditions of negligible spontaneous decay and $N\gg\alpha$, the intracavity Fano factor depends linearly on the zero-frequency normalized photocurrent spectral density [15], $\mathcal{S}=2\mathcal{F}-1$. This relation does not hold in general.

At some Fourier frequency Ω, the generalized rate equations read [2]

$$i\Omega\Delta m=\Delta R-\Delta Q, \tag{19a}$$

$$i\Omega\Delta n_0=\Delta D-\Delta J, \tag{19b}$$

$$i\Omega\Delta n_1=\Delta R+\Delta S-\Delta D, \tag{19c}$$

$$i\Omega\Delta n_2=\Delta U-\Delta R-\Delta S, \tag{19d}$$

$$i\Omega\Delta n_3=\Delta J-\Delta U. \tag{19e}$$

Equations (6), (7), (9), and (19) are solved for ΔQ. The formula for the light spectral density S is the same as Eq. (11) except that the coefficients $\tilde{c}_z$ are complex and frequency dependent,

$$\mathcal{S}(\Omega)=\frac{1}{\alpha m}\sum_{z\in\{j,d,u,q,r,s\}}\tilde{c}_z(\Omega)\tilde{c}_z^*(\Omega)\sigma_z, \tag{20}$$

where the Langevin "forces" σ_z are still given in Eq. (12).

After rearranging, Eq. (20) gives the spectral density in the form of the ratio of two polynomials of degree 4 in Ω^2 in the numerator and denominator. $\mathcal{S}(\Omega)$ tends to unity (shot-noise level) at high frequencies.

Figure 4 shows that $\mathcal{S}$ reaches its minimum value at $\Omega=0$. When spontaneous decay from the upper working level may be neglected, light is always sub-Poissonian and the lowest $\mathcal{S}$ value occurs when $P/p_d=1/2$. Spontaneous decay from the upper working level is inconsequential until $\gamma/p_d\approx 3\ 10^{-2}$. The light statistics ceases to be sub-Poissonian when $\gamma/p_d>0.3$.

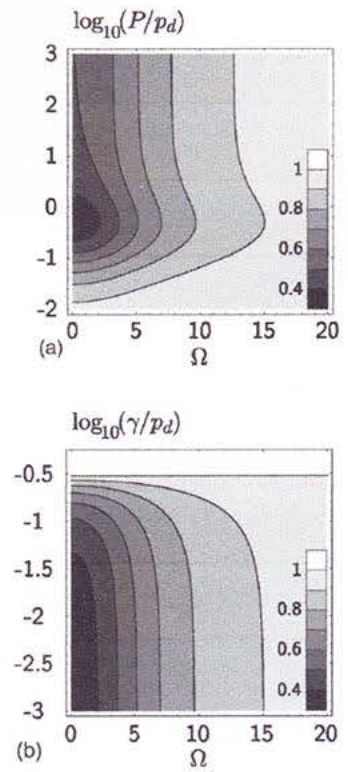

FIG. 4. Contour plots of normalized photocurrent spectra $\mathscr{S}$ as a function of the normalized Fourier frequency Ω and for incoherently pumped four-level atomic lasers. Parameters are $N=10^5$ atoms, $\alpha=6.32$, $p_d=632$, $p_u=316$. (a) Dependence of $\mathscr{S}(\Omega)$ on P/p_d with $\gamma=0$. (b) Dependence of $\mathscr{S}(\Omega)$ on γ/p_d with $P=316$.

The intracavity photon statistics is characterized by the Fano factor $\mathscr{F}=\langle\Delta m^2\rangle/\langle m\rangle$, the variance of m being obtained by integration over the frequency of the spectral density of Δm. The Fano factor of an incoherently pumped four-level atomic laser has been calculated as a function of the pump and spontaneous decay rates. The results fully agree with those previously reported by Koganov and Shuker [29].

IV. CONCLUSION

We have considered optically pumped four-level and three-level atomic lasers in resonant single-mode cavities. The light statistics has been obtained from a simple rate-equation approach, using both a Monte Carlo simulation and an analytical method based on linearization. The emitted light may be sub-Poissonian as was previously observed by many authors. Whenever a comparison is made, exact agreement with the previous results is noted. In the case of coherently-pumped three-level atomic lasers, our results are new to best of our knowledge. When the assumptions of negligible spontaneous decay and large atom numbers are not made, the results presented in this paper for the internal and external field statistics are new.

For practical reasons, Monte Carlo simulations were restricted to $N\approx 100$ atoms. Because the analytical formulas, obtained through the use of symbolic calculus, are lengthy they were not written down in the paper. However, they were employed to determine the conditions under which the spectral density of the photocurrent reaches its minimum value. For example, we found that when spontaneous decay from the upper working level may be neglected, three-level atomic lasers may deliver light with fluctuations at half the shot-noise level. Four-level atomic lasers may deliver light with fluctuations at one-third of the shot-noise level. The photocurrent noise decreases further and tends to zero, under ideal conditions, when the number of levels becomes large [14]. The present theory may be viewed as a building block to be incorporated in a fully realistic laser-noise theory accounting for phase noise and spatial and spectral inhomogeneities [26].

ACKNOWLEDGMENTS

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[1] K. An, J.J. Childs, R.R. Dasari, and M.S. Feld, Phys. Rev. Lett. **73**, 3375 (1994).
[2] J. Arnaud, Opt. Quantum Electron. **27**, 63 (1995).
[3] J. Arnaud, Opt. Quantum Electron. **34**, 393 (2002).
[4] This is a situation of thermodynamic, but not of *thermal*, equilibrium. To avoid a confusion, let us note that Rice and Carmichael [7] call "thermodynamic limit" the large spontaneous decay rate limit, an entirely different situation.
[5] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, 1983).
[6] Phase noise may be obtained through the introduction of quadrature *classical* Langevin forces, even when the medium is inhomogeneous broadened [33]. Spatial inhomogeneities were treated earlier [26] with the help of a circuit model. In that early paper, a simplified modeling of the active regions was employed for the sake of illustration. The present four-level atom treatment could, at a later stage, be incorporated into the Arnaud and Estéban [26] circuit theory in order to achieve a realistic treatment of lasers in which the regions with gain and with loss are spatially separated.
[7] P.R. Rice and H.J. Carmichael, Phys. Rev. A **50**, 4318 (1994).
[8] The "1" in the Einstein expression $m+1$ of stimulated atomic decay is often described as expressing "spontaneous emission in the oscillating mode." This interpretation makes sense for a particular operator ordering but is not mandatory. Emission and absorption transition rates are *both* equal to the field energy $m+1/2$ at jump time, the latter being defined from an arithmetic average of the field energy before and after a jump.
[9] M.O. Scully and W.E. Lamb, Jr., Phys. Rev. **159**, 208 (1967).
[10] A.M. Khazanov, G.A. Koganov, and E.P. Gordov, Phys. Rev. A **42**, 3065 (1990).
[11] T.C. Ralph and C.M. Savage, Phys. Rev. A **44**, 7809 (1991).

[12] H. Ritsch, P. Zoller, C.W. Gardiner, and D.F. Walls, Phys. Rev. A **44**, 3361 (1991).
[13] H. Ritsch and P. Zoller, Phys. Rev. A **45**, 1881 (1992).
[14] H.-J. Briegel, G.M. Meyer, and B.-G. Englert, Phys. Rev. A **53**, 1143 (1996).
[15] Y.M. Golubev and I.V. Sokolov, Zh. Eksp. Teor. Fiz. **87**, 408 (1984) [Sov. Phys. JETP **60**, 234 (1984)].
[16] S. Machida, Y. Yamamoto, and Y. Itaya, Phys. Rev. Lett. **58**, 1000 (1987).
[17] R.B. Levien, M.J. Collett, and D.F. Walls, Phys. Rev. A **47**, 5030 (1993).
[18] C. Zhu, Phys. Rev. A **48**, 3930 (1993).
[19] M.I. Kolobov, L. Davidovich, E. Giacobino, and C. Fabre, Phys. Rev. A **47**, 1431 (1993).
[20] F. Haake, M.I. Kolobov, C. Fabre, E. Giacobino, and S. Reynaud, Phys. Rev. Lett. **71**, 995 (1993).
[21] Y.M. Golubev, Theor. Math. Phys. **109**, 1437 (1996).
[22] H. Ritsch, M.A.M. Marte, and P. Zoller, Europhys. Lett. **19**, 7 (1992).
[23] L. Davidovitch, Rev. Mod. Phys. **68**, 127 (1996).
[24] T.C. Ralph and C.M. Savage, Quantum Opt. **5**, 113 (1993).
[25] L. Chusseau, J. Arnaud, and F. Philippe, Opt. Commun. **213**, 325 (2002).
[26] J. Arnaud and M. Estéban, IEE Proc.-J: Optoelectron. **137**, 55 (1990).
[27] D.T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic Press, San Diego, 1992).
[28] D.P. Landau and K. Binder, *A Guide to Monte Carlo Simulations in Statistical Physics* (Cambridge University Press, Cambridge, 2000).
[29] G.A. Koganov and R. Shuker, Phys. Rev. A **63**, 015802 (2000).
[30] L. Chusseau and J. Arnaud, Opt. Quantum Electron. **34**, 1007 (2002).
[31] A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York, 1991).
[32] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C* (Cambridge University Press, Cambridge, 1992).
[33] J. Arnaud, Quantum Semiclassic. Opt. **9**, 507 (1997).