PULSE PROPAGATION IN OPTICAL FIBERS

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INTRODUCTION

In this paper, we have no attempt to present some results which, basically, have not been known before. What we wish to emphasize is the analogy that exists between space and time concepts, pulse spreading in dispersive media being analogous to beam diffraction.

TIME-HARMONIC PLANE WAVES

Let us first consider a time-harmonic plane wave propagating in the x-z plane. The field has the form

$$\psi(x,z,t) = \cos(k_x x + k_z z - \omega t)$$

$$= \text{Real Part of } \{\exp[i(k_x x + k_z z)] \exp(-i\omega t)\} \qquad (1)$$

to within arbitrary amplitude and phase factors. The numbers k_x, k_z can be considered the components of a wave vector $\vec{k}$. The angular frequency ω is taken to be a constant here. The crests of the field are clearly given at $t = 0$ by the straight lines

$$k_x x + k_z z = 2m\pi \quad ; \quad m = 0, \pm 1, \pm 2 \ldots \qquad (2)$$

shown on top of Fig.1 (a), these lines being perpendicular to the $\vec{k}$ vector.

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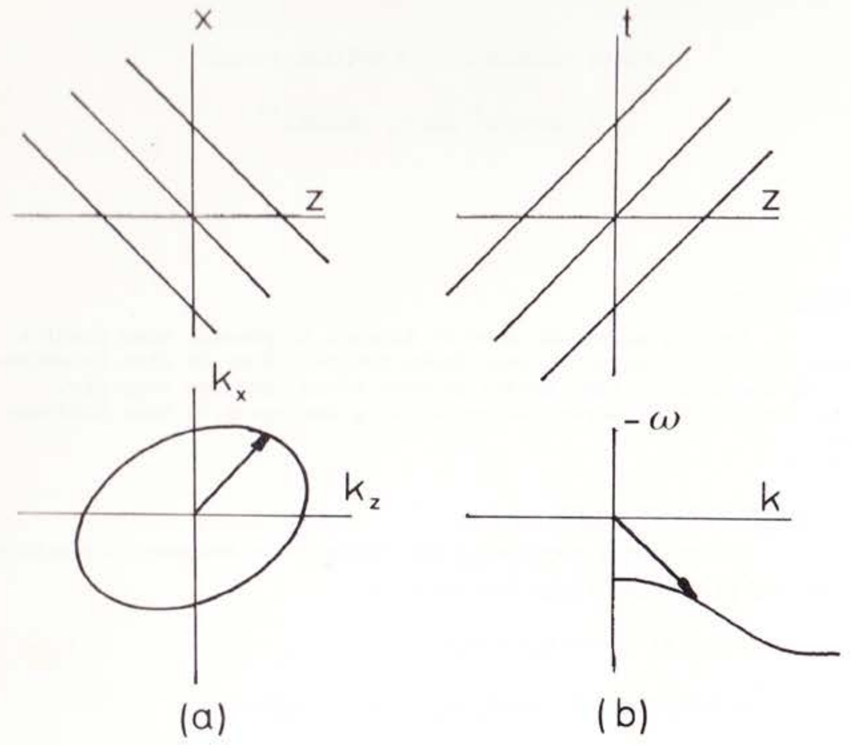

Fig.1

The distance between adjacent crests in the x direction at a given time is $\lambda_x = 2\pi/k_x$, and, similarly $\lambda_z = 2\pi/\lambda_z$. We define $k = |\vec{k}| = (k_x^2 + k_z^2)^{1/2}$ and the wavelength in the medium (not to be confused with the free-space wavelength λ_o) is $\lambda = 2\pi/k$. If λ depends on the direction of $\vec{k}$, the medium is called anisotropic. In isotropic media the curve described by the tip of the $\vec{k}$ vector is a circle. The nonzero curvature of that circle is responsible for beam diffraction as we shall see.

Let us now consider phenomena depending on time (t) and one spatial coordinate (z). The field has the form

$$\psi(z,t) = \cos(kz - \omega t) = \text{Real Part of } \exp[i(kz - \omega t)] \qquad (3)$$

We shall omit the subscript z on k when there is no risk of ambiguity. The crests are given by the straight lines

$$kz - \omega t = 2m\pi \quad ; \quad m = 0, \pm 1, \pm 2 \ldots \qquad (4)$$

shown on top of Fig.1(b). If T denotes the wave period, that is the interval of time separating two successive maxima of the field, then $\omega = 2\pi/T$. Similarly, $k = 2\pi/\lambda$. Clearly from Eq.(3), the wave crests in space-time are perpendicular to the $\vec{k}$ vector with components k, $-\omega$. The $\vec{k}$ curve at the bottom of Fig.1b is a straight line going through the origin when the medium is non dispersive, k being proportional to ω, or $k = (\omega/c)n$, with n a constant. Alternatively one can say that the phase velocity $v_\phi = c/n$ is independent of frequency. But a general medium is dispersive. For example, if we restore the dependence of ψ on the x coordinate and maintain k_x a constant (this is the case when a plane wave is launched on a transmission grating of period p, then $k_x = \pm 2\pi/p$, and for a metallic waveguide or an oversized optical fiber of width d : $k_x = \pi/d$), k_z is no longer proportional to ω. In that sense, free-space can be viewed as dispersive. Indeed, in free space

$$k_z = [(\omega/c)^2 - k_x^2]^{1/2} \; , \; k_x = \text{constant.} \qquad (5)$$

implies a nonlinear dependence of k_z on ω.

An optical fiber (of axis z) is dispersive also because silica glass is a Lorentz medium that can be modeled by microscopic oscillators in the ultraviolet (UV) range and in the infrared (IR) range. Low-loss propagation occurs in between (from, say, $\lambda_o \approx 0.8$ µm to 1.7 µm). In such a medium the refractive index squared is, neglecting losses,

$$n^2 = 1 - \frac{A_1}{\omega^2 - \omega_1^2} - \frac{A_2}{\omega^2 - \omega_2^2} \qquad (6)$$

where A_1, A_2 are oscillator strengths and ω_1, ω_2 are resonant frequencies in the UV and IR, respectively.

We have the correspondence

$x, z \longleftrightarrow z, t$
$k_x, k_z \longleftrightarrow k$ (or k_z), $-\omega$
$\lambda_x, \lambda_z \longleftrightarrow \lambda$, T

(ω fixed) (k_x fixed)

RAY TRAJECTORIES

Next, let us consider the average direction, or speed, of wave packets. These quantities involve now the slopes of the $\vec{k}$ curves shown at the bottom of Fig.1, in the neighborhood of the carrier wavevector $\vec{k} = \vec{k}_o$.

In the x-z space domain, we need consider transversely limited beams whose widths comprise however many wavelengths and we are concerned with the direction in which the beam has maximum energy. This is the ray direction. In the space-time domain, we are considering wave packets of duration much larger than the period T. The average speed of a wave packet is called the group velocity, and its trajectory in space-time is called a worldline (or space-time ray).

As we shall see in the next section beams and wave packets in general spread out and distort about their average motion, and one may wonder if the wave packet center is a well-defined concept. This is so because, as we make the propagation length L longer and longer, the transit time $t = L/v_g$ increases in proportion to length, while the pulse spreading increases only in proportion of $\sqrt{L}$ if we are careful to increase the input pulse duration in proportion to $\sqrt{L}$ in order to minimize its frequency spectrum. Thus, the accuracy with which one can measure the group velocity can be increased without limit, in spite of the wavepacket deformation.

To summarize this section, we have the correspondance :

ray trajectory in space $\longleftrightarrow$ world line
space-space $\longleftrightarrow$ space-time

Finally, we shall be concerned with the distortion of a beam about its average path and the spreading of a wave packet about its world line, with the correspondence

diffraction $\longleftrightarrow$ pulse spreading
space-space $\longleftrightarrow$ space-time

These latter effects are dependent on the curvature of the k-curves shown at the bottom of Fig.1. We could go on step further and consider geometrical aberrations in space and time, but this will not be discussed here.

Because we shall be concerned only with a small neighborhood of the ω, k curve, centered at ω_o, k_o, we restrict ourselves to the following representation of the $k(\omega)$ curve

$$k = k_o + a(\omega - \omega_o) + \frac{1}{2} b(\omega - \omega_o)^2 \qquad (8)$$

were $a = dk/d\omega$, $\omega = \omega_o$ and $b = d^2k/d\omega^2$, $\omega = \omega_o$. For a given material and a given carrier frequency (or free-space wavelength λ_o), k_o, a and b are fixed quantities. It is useful to introduce in place of k_o, a, b dimensionless quantities n, D and M as was done in Ref.1

$$k = (\omega/c)n$$
$$d = D - 1 = (\omega/k)\, dk/d\omega - 1$$
$$M = (\omega^2/k) d^2k/d\omega^2 = (\lambda_o^2/n)(d^2n/d\lambda_o^2) \qquad (9)$$

where all quantities are evaluated at $\omega = \omega_o$.

For silica at $\lambda_o = 0.9$ µm, for example, we have :

$$n = 1.45 \quad ; \quad d = 8.8 \times 10^{-3} \quad ; \quad M = 12 \times 10^{-3}$$

As λ_o increases, the d factor increases slightly, while M goes to zero around 1.3 µm, and then becomes negative. For other materials of practical use, the wavelength at which M is zero may be larger, e.g., $\lambda_o = 1.7$ µm for pure germania. The physical significance of these results will be appreciated later.

OPTICAL LENGTH OF A SPACE-TIME RAY

We wish to evaluate the field generated at z and time t by a short pulse (not a Dirac function) with carrier frequency ω_o, emitted at z = 0 and t = 0. Within the approximation of Eq.(8), this is the Van Vleck propagator (1928) discussed in Ref.1 for square-law anisotropic media. We restrict ourselves here to homogeneous media, and we will give only the rapidly varying phase term $\exp[iS(z,t)]$. It is understood that, at the end of the calculations, one must take the real part of the quantity obtained, since the field, being a measurable quantity, must be real at some z, t point (in many optical experiments, however, it is sufficient to know the complex field, or more precisely, the analytic signal).

Because the medium is homogeneous and time-invariant, the world lines are straight with slope

$$\frac{t}{z} = \frac{dk}{d\omega} = a + b\,(\omega - \omega_o) \qquad (10)$$

where a and b are the constants introduced in Eq. (8). This slope is the reciprocal of the group velocity v_g. It takes the value a precisely at $\omega=\omega_o$. Unless b = 0, the world lines fan out from the origin z = t = 0. The initial pulse has a frequency spectrum centered about $\omega=\omega_o$ with a width reciprocal to the (as yet unspecified) pulse duration. The frequency that reaches some specified point z, t in space-time is clearly, from Eq. (10)

$$\omega=\omega_o + (t/z - a)/b \qquad (11)$$

Because the pulse spectrum is fairly narrow, most of the pulse energy is to be found whenever t/z is close to a (reciprocal of the group velocity).

The spatial analog of the situation depicted above are rays emitted from a small spot (not quite a point source). The ray directions are normal to the surface of wave normals $x/z = dk_x/dk_z$. This is only for isotropic media that we have $x/z = k_x/k_z$ (rays directed along the $\vec{k}$ vectors).

Next, we calculate the phase shift [see Eq. (3)]

$$S(z,t) = k(\omega)\, z - \omega t \qquad (12)$$

Substituting $\omega-\omega_o$ from Eq. (11) into Eq. (8) and Eq. (12), we obtain, after rearranging,

$$- S = \omega_o t - k_o z + \frac{1}{2} (t - az)^2/bz \qquad (13)$$

In space-space, at some fixed frequency ω, $S/2\pi$ (if this is an integer) corresponds physically to the number of crests of the field from x = z = o to the x, z point, at some given time. Alternatively, we can view the S(x,z) = constant curve as the wavefront emitted by the radiating spot. This curve differs vastly from the surface of wave normals ($\vec{k}$ curve), except if the latter is circular.

In space-time $S/2\pi$ (if this is an integer) is the number of crests from t = z = 0 to the t, z point. (For the analogy to be complete, one should restore the x coordinate and keep k_x a constant, in the same way as we kept ω a constant previously).

In connection with the hamiltonian - lagrangian formalism developed in Ref.1, note incidentally that Eq. (10) is one of the two Hamilton's equations, the Hamilton characteristic function being taken as $\omega = \omega(k)$. S in Eq.(13) is the product L(z, dz/dt) x t, L being the Lagrangian function, easily deducible from the Hamiltonian function.

Note also that the result in Eq. (13) can be obtained alternatively by expanding ω up to second order in $k - k_o$ (instead of the opposite), provided z is replaced in the denominator of the last term by t/a. This is a permissible approximation because the power is significant only in the neighborhood of z = t/a.

REPRESENTATIONS OF PULSE SPREADING

It is often pointed out that pulse spreading in dispersive media is analogous to beam diffraction. In particular, a gaussian pulse of carrier frequency ω_o and initial duration σ_o (defined at the 1/e point of the field intensity) spreads out after a propagation length z to σ given by

$$\sigma^2 = \sigma_o^2 + b^2 z^2/\sigma_o^2 \qquad (14)$$

where $b = d^2k/d\omega^2$, $\omega = \omega_o$, as before. This result can be obtained by multiplying the pulse spectrum

$$\exp\left[- \frac{1}{2} \sigma^2 (\omega-\omega_o)^2\right] \qquad (15)$$

by the phase function $\exp\left[ik(\omega)z\right]$ and performing a Fourier transform back to the time-dependent field. It is quicker to use the complex-ray representation of gaussian beams proposed by one of us in 1968 for isotropic[3,4] media and shortly there after for anisotropic media[1].

The half-width of a gaussian beam propagating in a medium with wave number k_o on the other hand spreads out according to the law[1]

$$\xi^2 = \xi_o^2 + z^2/k_o^2 \xi_o^2 \qquad (16)$$

whose similarity with Eq.(14) is obvious. The space-time analogy is of course not limited to gaussian beam pulses. More generally, one can set up a vertical axis $t - z/v_g$, where v_g is the group velocity, and describe pulse spreading as if it were a diffraction problem in medium with wave number b^{-1}. This procedure, however, is incomplete and provides only pulse envelopes. The reason why the above description is uncomplete is that pulse spreading is analogous to diffraction in anisotropic media, while Eq.(16) as written, is applicable only to isotropic media. It takes a coordinate transformation to go from anisotropic to isotropic media. The gaussian beam described by Eq.(16) for example, has zero average transverse spatial frequency at z = 0, while pulses have always non zero carrier frequencies. It is not sufficient to subtract z/v_g from t to take that fact into account.

It is however possible to find coordinates ξ, ζ such that the expression for the phase shift S in Eq.(13) be equal to $\varpi(\xi^2 + \zeta^2)^{1/2}$, as it would be the case for isotropic media, in the neighborhood of the group delay, up to second order. We will give only the result : one must set on the vertical axis, instead of t the quantity

$$\xi = t - \left(\frac{\cos^2\Theta}{v_g} + \frac{\sin^2\Theta}{v_\phi}\right) z \qquad (17)$$

On the horizontal axis, instead of z, one must set

$$\zeta = \left(\frac{1}{v_\phi} - \frac{1}{v_g}\right) \sin\Theta \cos\Theta\, z \qquad (18)$$

and the effective angular frequency $\overline{\omega}$ is

$$\overline{\omega} = \omega_o/\sin\Theta \qquad (19)$$

In these expressions, Θ defines the direction of a pulse at the mean group velocity $v_g = a^{-1}$; $v_\phi = \omega_o/k_o$ is the phase velocity. We have

$$\tan\Theta = (M/d)^{1/2} \qquad (20)$$

if we introduce the dimensionless parameters M and d. This representation is restricted to positive M/d ratios (in silica, $\lambda_o < 1.3\ \mu m$). Note, however, that the field modulus is unchanged if M is changed to -M (as an example see Eq.(14) where only b^2 enters). This observation extends the range of validity of the proposed representation. For the values of n, d and M given earlier (Silica at $\lambda_o = 0.9\ \mu m$), we find from Eqs.(17) to (20)

$$\Theta = 50°$$
$$\xi = t - 1.45\, z/c$$
$$\zeta = 0.0063\, z/c$$
$$\overline{\omega} = 435 \text{ Terahertz}$$

where c is the velocity of light in free space. Using these coordinates, pulse spreading can indeed be accurately described if it were beam diffraction in free space.

GENERALIZATIONS

The complex ray representation of gaussian pulses alluded to before enables us to deal with media that are not only dispersive but also have quadratic spatial temporal variations of the refractive index. For example, the pulse spreading of tubular modes measured in Ref. 5 by interferometric techniques can be dealt with accurately using such a representation.

Nonlinear (Kerr) effects can be dealt with also, and solitary waves discovered, with the above formalism, provided one assumes that the refractive index is a logarithmic (rather than linear) function of the optical intensity. Whether this assumption is academic or has some physical significance remains to be seen.

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