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LETTERS AND COMMENTS

Pulsating Gaussian wavepackets and complex trajectories

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Abstract. Pulsating Gaussian wavepackets are constructed from complex trajectories.

Gaussian wavepackets illustrate a basic feature of quantum mechanics, namely the fact that any reduction in particle-position uncertainty increases the momentum uncertainty. Likewise, Gaussian optical or acoustical beams illustrate the basic concept of diffraction in free space or square-law media. The present letter offers a simple picture of Gaussian wavepacket (or Gaussian beam) evolution. Suggestions are made at the end of the paper for employing that picture in a classroom.

Consider a particle of mass m and position x subjected to a restoring force $-Kx$. The equation of motion of $x = q(t)$ reads

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \omega^2 q = 0 \qquad (1)$$

where the oscillation (angular) frequency is $\omega = \sqrt{K/m}$. The purpose of this paper is to show that the construction of Gaussian wavepackets from complex solutions of (1) provides a more intuitive picture than do formal solutions, such as those given in [1] and the references therein.

A complex trajectory $q(t)$ is defined as $q'(t) + \mathrm{i}q''(t)$, where $q'(t)$ and $q''(t)$ denote two real solutions of (1) not proportional to one another. One can show that

$$\psi(x, t) = \frac{1}{\sqrt{q(t)}} \exp\left[\mathrm{i}k(t)\frac{x^2}{2q(t)}\right] \qquad (2)$$
$$\hbar k \equiv \hbar(k' + \mathrm{i}k'') \equiv m\frac{\mathrm{d}q}{\mathrm{d}t}$$

(where $\hbar$ denotes the Planck constant divided by 2π, and the arguments t of $q(t)$ and $k(t)$ are occasionally omitted) is a solution of the Schrödinger equation for the potential $V(x) = Kx^2/2$

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}Kx^2\right)\psi. \qquad (3)$$

To prove this, it suffices to take the logarithm of (2), differentiate once with respect to t and twice with respect to x, substitute in (3) and rearrange.

The solution (2) was first obtained in 1968 for the analogous problem of Gaussian beam propagation through inhomogeneous media [2]‡, the medium axial coordinate z playing the role of time t, and the electric field $E(x, z)$ the role of the wavefunction $\psi(x, t)$. Equation (2) is then easy to interpret when q is a real function of time: in the argument of the exponential, $x^2/2q$ expresses the departure of a circular wavefront from the x-axis. On the other hand, the pre-factor $1/\sqrt{q}$ follows from the law of energy conservation that entails that $|E|^2 q$ should be a constant. The singularity at $q = 0$ is removed when complex values are assigned to q, and Gaussian beams are then obtained.

The real and imaginary components $q'(t)$ and $q''(t)$ of $q(t)$ may be viewed as the two transverse (x, y) components of a real ray trajectory. Accordingly, Gaussian wavepackets may be simulated with the help of light rays, launched off-axis, and possibly refracted by ordinary lenses (see figure 2.4 of [2]).

‡ In reference [2], see especially figures 1.9, 1.10 and 2.4.

The authors of a recent text-book [3] (see page 45) give the Gaussian solution in a form resembling (2) but with the argument of the exponential term written as $-(m/2\hbar)C(t)x^2$, where $C(t)$ is defined as a solution of the nonlinear differential equation: $\mathrm{d}C/\mathrm{d}t = \mathrm{i}\omega^2 - \mathrm{i}C^2$. They do not point out that the function $q(t)$ defined by $\mathrm{i}C = (\mathrm{d}q/\mathrm{d}t)/q$ obeys the much simpler equation (1).

It follows from (2) that the probability density for finding the particle at x at time t is a Gaussian

$$|\psi(x,t)|^2 \propto \exp\left[-\left(\frac{x}{\xi(t)}\right)^2\right] \qquad (4)$$

where $\propto$ denotes proportionality, and

$$\frac{1}{\xi(t)^2} = \mathrm{Im}\left(\frac{k}{q}\right) = \frac{q'k'' - q''k'}{q'^2 + q''^2} \qquad (5)$$

where Im represents the imaginary part of its argument. It is thus natural to normalize the complex trajectory $q(t)$ by the condition

$$q'k'' - q''k' = 1. \qquad (6)$$

Differentiating the left-hand-side of (6) with respect to t and using (1), one observes that this condition holds at all times if it holds at $t = 0$. The width $\xi(t)$ of the Gaussian wavepacket defined in (5) is then simply equal to the modulus of $q(t)$, that is: $\xi(t)^2 = q'(t)^2 + q''(t)^2$. It suffices to find two real solutions $q'(t)$ and $q''(t)$ of (1), and normalize them according to (6), to obtain $\xi(t)$. For example

$$\begin{aligned} q'(t) &= \xi(0)\cos(\omega t) \\ q''(t) &= \frac{\hbar}{m\omega\xi(0)}\sin\omega t \end{aligned} \qquad (7)$$

give

$$\begin{aligned} \xi(t)^2 = {} & [\xi(0)\cos(\omega t)]^2 \\ & + \left[\frac{\hbar}{m\omega\xi(0)}\sin(\omega t)\right]^2. \end{aligned} \qquad (8)$$

This solution coincides with equation (8) of [1] if the changes of notation $\Delta x \to \xi/\sqrt{2}$ and $\alpha \to 1/\xi(0)$ are made.

Alternatively, the complex trajectory may be written as

$$\begin{aligned} q(t) &= a^+\mathrm{e}^{\mathrm{i}\omega t} + a^-\mathrm{e}^{-\mathrm{i}\omega t} \\ 2a^\pm &\equiv \xi(0) \pm \frac{\hbar}{m\omega\xi(0)} \end{aligned} \qquad (9)$$

in which case $\xi(t)$ may be viewed as the length of the sum of two constant-length vectors rotating in opposite directions in the complex q-plane. The wavepacket pulsates unless $a^- = 0$.

As a further description, consider an ellipse centred at the origin rotating at angular frequency ω in the two-dimensional phase space q, $\mathrm{d}q/\mathrm{d}(\omega t)$. The projection of that ellipse on the x-axis, representing the width of the wavepacket, pulsates at twice the angular frequency ω. This phase-space picture is often employed in 'quantum optics' ([4], see especially the cover page and figure 3.3).

Because the complex-trajectory picture of Gaussian wavepackets is algebraically simple and easy to visualize, students may appreciate being presented this type of solution before going into more formal methods. I suggest that the evolution in time of a *free* particle be treated first. In that case, complex rays may be represented as straight lines rotating about some non-intersecting axis, as discussed above. It is well known in geometry that such a rotation generates a hyperboloid of revolution (the profile of Gaussian wavepackets). The next step would consist of letting the straight line (a geometrical optics ray) be refracted by a single converging lens, and subsequently by a sequence of closely spaced weakly converging lenses. The latter experiment illustrates Gaussian wavepacket pulsation.

References

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[4] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)