PULSE BROADENING IN MULTIMODE GRADED-INDEX FIBRES

Indexing terms: Fibre optics, Optical waveguides

A closed-form expression is obtained for pulse broadening in graded-index fibres with $k^2(r) = 1 - r^2 + \varepsilon_2 r^4 + \ldots + \varepsilon_n r^{2n}$. Pulse broadening for $k^2(r) = 1 - r^2 + 0{\cdot}615 r^4 + 70 r^6$ and $r < 0{\cdot}1$ is 12 times smaller than for square-law fibres if material dispersion is neglected.

When material dispersion can be neglected, the broadening of optical pulses propagating in multimode graded-index fibres can be evaluated by comparing the optical lengths of rays excited by the source. Recently, Steiner[1] gave a solution based on a vector differential equation for the ray trajectory. The approach proposed in this letter is based on a simpler scalar equation. A closed-form solution is obtained for small, but arbitrary, deviations from a square-law profile.

The Hamilton equations for light rays $x(z)$, $y(z)$ in an isotropic z-invariant medium with wavenumber $k(x, y)$ are[2]

$$\left.\begin{aligned} dx/dz &= k_x/k_z \\ dy/dz &= k_y/k_z \\ dk_x/dz &= (1/2k_z)\partial k^2/\partial x \\ dk_y/dz &= (1/2k_z)\partial k^2/\partial y \end{aligned}\right\} \qquad (1)$$

The ratio of the time of flight along a ray to the corresponding time along the axis is[2]

$$T = (k_0/k_z)\langle \partial k^2/\partial\omega^2\rangle/(dk_0{}^2/d\omega^2) \qquad (2a)$$

where $k_0 \equiv k(0, 0)$ and $\langle\ \rangle$ denotes an average over a ray period. When material dispersion can be neglected, eqn. $2a$ reduces to

$$T = \langle k^2\rangle/k_0 k_z) \qquad (2b)$$

The axial component k_z of k is a constant of motion. From eqns. 1, we obtain

$$\tfrac{1}{2}k_z{}^2\, d^2(X+Y)/dz^2 = k^2 - k_z{}^2 + X\partial k^2/\partial X + Y\partial k^2/\partial Y \qquad (3)$$

where $X \equiv x^2$, $Y \equiv y^2$. If we integrate eqn. 3 over a ray period, the l.h.s. vanishes because $d(X+Y)dz$ assumes the same value at the limits of integration. Further, if we assume that k^2 is equal to $k_0{}^2$ plus a homogeneous function of degree α in X and Y, we have

$$X\partial k^2/\partial X + Y\partial k^2/\partial Y = \alpha(k^2 - k_0{}^2) \qquad (4)$$

In that case, a simple and exact expression for T is obtained:

$$T = \{(k_z/k_0) + \alpha(k_0/k_z)\}/(1+\alpha) \qquad (5)$$

Our result, eqn. 5, agrees with that given in Reference 3 for the special case

$$k^2 = 1 - (X+Y)^\alpha \equiv 1 - R^\alpha$$

where $R \equiv x^2 + y^2 \equiv r^2$.

Let us now consider a square-law medium ($\alpha = 1$)

$$k^2 = 1 - R \qquad (6)$$

The ray equation is easily solved. We obtain

$$R = \tfrac{1}{2}A(1+\theta) + \tfrac{1}{2}A(1-\theta)\cos(2z/k_z) \qquad (7)$$

where A denotes the square of the maximum radius of the ray and $\theta \equiv (L_z/A)^2$, where $L_z = xk_y - yk_x$ denotes the axial component of the ray angular momentum (the second constant of motion). For meridional rays, we have $\theta = 0$, and, for helical rays, $\theta = 1$. For later use, let us evaluate $\langle R^n\rangle$. Using the binomial expansion and the result

$$\langle \cos^m\rangle = m!2^{-m}\{(m/2)!\}^{-2} \qquad (8)$$

for even m, $\langle\cos^m\rangle = 0$ for odd m, we obtain

$$\langle R^n\rangle = n!2^{-n}A^n \sum_{m=0,2}^{n} \frac{(1+\theta)^{n-m}(1-\theta)^m}{2^m(n-m)!\{(m/2)!\}^2} \qquad (9a)$$

In particular,

$$\left.\begin{aligned} \langle R^2\rangle &= A^2(3\theta^2 + 2\theta + 3)/8 \\ \langle R^3\rangle &= A^3(1+\theta)(5\theta^2 - 2\theta + 5)/16 \end{aligned}\right\} \qquad (9b)$$

Let us now consider a perturbed square-law profile

$$k^2 = 1 - R + \sum_{n=2}^{N} \varepsilon_n R^n \qquad (10)$$

where the $\varepsilon_n R^{n-1}$ are of order ε. For circularly symmetric fibres, eqn. 3 becomes, after integration over a ray period,

$$0 = \langle k^2 - k_z{}^2 + R\partial k^2/\partial R\rangle \qquad (11)$$

Substituting eqn. 10 in eqn. 11, we obtain, for T in eqn. $2b$,

$$T = \tfrac{1}{2}k_z{}^{-1}\{k_z{}^2 + 1 + \sum_{n=2}^{N}(1-n)\varepsilon_n\langle R^n\rangle\} \qquad (12)$$

Eqn. 12 is exact. To 1st-order in ε, the zeroth-order approximation, eqn. 7, for R can be used in eqn. 12 and k_z can be expressed in terms of A and θ:

$$k_z{}^2 = 1 - A(1+\theta) \qquad (13)$$

Thus eqns. 9, 13 and 12 give a closed-form expression for the time of flight T of a pulse for any small deviation from a square-law profile.

Let the pulse broadening τ be defined as the maximum variation of T for $0 < \theta < 1$ and $0 < A < A_0$. For the square-law fibre in eqn. 6, we obtain $\tau = 0{\cdot}5A_0{}^2 = 20\times10^{-7}$ if $A_0 = 0{\cdot}002$. For $k^2 = 1 - R + \varepsilon_2 R^2$, we find that $T = 1$ for meridional rays ($\theta = 0$) when $\varepsilon_2 = 2/3$, in agreement with

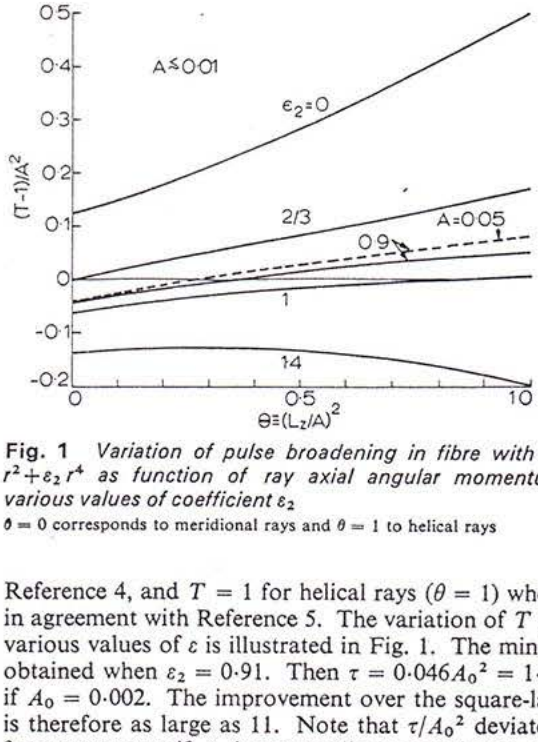

Fig. 1 *Variation of pulse broadening in fibre with $k^2 = 1 - r^2 + \varepsilon_2 r^4$ as function of ray axial angular momentum L_z for various values of coefficient ε_2*

$\theta = 0$ corresponds to meridional rays and $\theta = 1$ to helical rays

Reference 4, and $T = 1$ for helical rays ($\theta = 1$) when $\varepsilon_2 = 1$, in agreement with Reference 5. The variation of T with θ for various values of ε is illustrated in Fig. 1. The minimum τ is obtained when $\varepsilon_2 = 0{\cdot}91$. Then $\tau = 0{\cdot}046A_0{}^2 = 1{\cdot}84\times10^{-7}$ if $A_0 = 0{\cdot}002$. The improvement over the square-law profile is therefore as large as 11. Note that $\tau/A_0{}^2$ deviates slightly from a constant if A_0 is not small compared with unity. The broken line in Fig. 1 for $A_0 = 0{\cdot}05$ (corresponding to a total variation of refractive index of $2{\cdot}5\%$) deviates slightly from the plain line, obtained for $A_0 \lesssim 0{\cdot}01$. For $k^2 = 1 - R + \varepsilon_2 R^2 + \varepsilon_3 R^3$, and $A_0 = 0{\cdot}002$, the minimum τ is found for $\varepsilon_2 = 0{\cdot}615$ and $\varepsilon_3 = 70$: it is equal to $1{\cdot}62\times10^{-7}$. The improvement, compared with the case where $\varepsilon_3 = 0$, is therefore rather modest. It is interesting, however, that the optimum ε_2 is so sensitive to the choice of ε_3. It is conceivable that the absolute minimum of τ would be obtained for a strongly oscillating, ill-behaved, variation of k^2 with r. If this were the case, higher-order terms should be considered before a final conclusion can be reached concerning the optimum profile of a graded-index fibre. The method proposed in this work can also be used to investigate more complicated near-square-law fibres, such as the helical fibre[6] with $k^2 = 1 + x^2 - y^2 + \ldots$ in a rotating co-ordinate system.

Pulse broadening in square-law and linear-law fibres that have material dispersion is given in Reference 2. More generally, we may have

$$k^2(r) = k_0{}^2 - k_1{}^2 R + \sum_{n=2}^{N} k_n{}^2 R^n \qquad (14)$$

Proceeding as before, we obtain, for the pulse broadening,

$$T = (k_0/k_z)\left\{1-\tfrac{1}{2}(1-k_z^2/k_0^2)D_1 + \sum_{n=2}^{N}[D_n-\tfrac{1}{2}(n+1)D_1]\varepsilon_n\langle R^n\rangle\right\} \quad (15)$$

where we have defined the dispersion factors

$$D_n = (k_0^2\, dk_n^2/d\omega^2)/(k_n^2\, dk_0^2/d\omega^2) \quad . \quad . \quad . \quad . \quad (16)$$

that are unity in the absence of dispersion, $\varepsilon_n \equiv k_n^2/k_0^2$, and $\langle R^n\rangle$ is given in eqn. 9 with $\theta = (L_z/k_1 A)^2$. Thus a closed-form solution for pulse broadening in dispersive fibres that have small, but otherwise arbitrary, deviations from the square-law has been obtained.

J. A. ARNAUD

28th November 1974

Bell Laboratories Inc.
Crawford Hill Laboratory
Holmdel, NJ 07733, USA

References

1 STEINER, K. H.: *Nachrichtentech. Z.*, 1974, **27**, p. 250
2 ARNAUD, J. A.: *Bell Syst. Tech. J.*, 1974, **53**, p. 1599
3 GLOGE, D., and MARCATILI, E. A. J.: *ibid.*, 1973, **52**, p. 1563
4 LUNEBURG, R. K.: 'Mathematical theory of optics' (University of California Press, Berkely, 1964), p. 180
5 KAWAKAMI, S., and NISHIZAWA, J.: *IEEE Trans.*, 1968, **MTT-16**, p. 814
6 ARNAUD, J. A.: *Appl. Opt.*, 1972, **11**, p. 2514