Natural linewidth of semiconductor lasers

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Abstract: A general and simple expression for the natural linewidth of lasers with high, spatially inhomogeneous gain, such as semiconductor lasers, has recently been reported by the author in a short paper. In the present paper, the relation is applied to a number of circuits relevant to semiconductor injection lasers. Its significance is clarified by considering simple lumped circuits. It is further generalised to anisotropic materials which may be nonreciprocal, and the role of dispersion inside and outside the laser cavity is discussed. The theory is restricted to single-mode operation (well above threshold operation), but saturation is neglected.

1 Introduction

It is well known that transverse inhomogeneities in gain enhance the natural linewidth of high-gain lasers, by a factor $K \gg 1$ (see References 1 and 2). By 'natural linewidth' we mean the nonzero linewidth which results from spontaneous emission in the active medium. The theories in References 1 and 2, however, are essentially 1-dimensional. A formula derived from first principles is given here which generalises previous results to arbitrary 3-dimensional geometries, and makes them more precise [3]. That formula shows that a K-like factor exists also in the longitudinal direction. The fact that the Shawlow-Townes (ST) formula is not applicable when the mirror reflectivity is not close to unity has been recognised earlier [4, 5]. Although this fact can sometimes be overlooked [6, 7] for conventional lasers (error $\sim 10\%$), it is important to take it into account in the case of reduced reflectivity lasers. The departure of this longitudinal K-like factor from unity is not as large as in transverse directions, but it is nevertheless significant when the mirror power reflectivities are much smaller than unity. When $P_R = 0.01$, for example, the laser linewidth Δf is 4.6 times the value calculated from the ST formula. The fact that Petermann's K factor and the longitudinal linewidth enhancement are derived from the same basic formula was, to our knowledge, first pointed out in Reference 3. A recent paper by Henry [8] goes in the same direction. However, Henry considers only configurations that are uniform along the laser axis between two partially reflecting mirrors. To obtain a comprehensive 3-dimensional formula, it is essential to include the detector of radiation in the laser cavity.

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If the effect of saturation is to be treated consistently, it is essential to introduce some nonuniformity along the laser structure because the field intensity is larger near the end mirrors and this results in a reduced carrier density there, when the series resistance is not negligible. A change in carrier density affects, in turn, the complex medium permittivity. For the sake of simplicity, saturation is neglected in the present paper. Therefore, formulas assuming full saturation such as the ones given, for example, by Henry [6] should be multiplied by a factor of 2, because intensity fluctuations are not suppressed. The α factor will not appear because the carrier density is independent of time in the unsaturated regime.

Single-spatial-mode (both transverse and longitudinal) operation is assumed. One may wonder whether it is consistent to neglect saturation and at the same time consider single-mode operation. Indeed, the former assumption is usually considered valid below threshold and the latter well above threshold. However, circumstances may be found in which both assumptions are in fact valid simultaneously. This may be the case, for example, when electrons and holes recombine mainly nonradiatively. Then the number of electrons does not fluctuate much as a result of the optical field intensity fluctuations. Another case is when the voltage across the junction and therefore, approximately, the energy difference between the quasi-Fermi levels, is held independent of time with the help of the appropriate electronics. However, saturation is in fact important for most semiconductor lasers driven by a constant current well above threshold. Saturation is neglected here for simplicity, as a first step toward a full solution.

Under those assumptions, the laser field can be viewed as amplified spontaneous emission. Spontaneous emission is modelled by electrical currents $J(r)$ randomly oriented and δ-correlated in space. The averaged quantity $\langle \boldsymbol{J} \cdot \boldsymbol{J}^* \rangle$ is proportional to hf, where h is Planck's constant and f the optical frequency, and to the negative imaginary part of the medium permittivity ε. This representation is discussed in detail, for example, in Reference 9. The rest of the calculation is based on linear electromagnetism.

Let us consider a cavity with perfectly conducting walls which encloses both the laser and the detector of radiation. The permittivity ε and permeability μ of the medium are, in general, complex scalar functions of space $\boldsymbol{r} = x, y, z$ and frequency f. The imaginary part of ε is split into $\varepsilon_1 - \varepsilon_2$, where ε_1 represents stimulated absorption (including absorption by the detector) and ε_2 represents stimulated emission. Conditions well above threshold are considered in which the resonating field $\boldsymbol{E}$, $\boldsymbol{H}$, is almost in a single spatial mode. Under those conditions, the product of laser (full half-power) linewidth Δf and dissipated power P ($P = 2P_o$ is the laser output power if the internal losses are neglected and P_o is the power per facet in a symmetrical configuration) is given by

$$\Delta f P = 4(hf/2\pi)\left| \int \sigma |E|^2 \, dV \Big/ \int (\varepsilon' E^2 - \mu' H^2) \, dV \right|^2 \quad (1)$$

where $\sigma = 2\pi f \varepsilon_1$ represents the medium conductivity. The following have been defined: $\varepsilon' = \partial(f\varepsilon)/\partial f$, $\mu' = \partial(f\mu)/df$. These quantities would be equal to ε and μ, respectively, if the medium were nondispersive, but dispersion is important in many applications. For any vector $\boldsymbol{E}$, $|\boldsymbol{E}|^2 = \boldsymbol{E} \cdot \boldsymbol{E}^*$ where the star indicates the complex conjugate and the dot a scalar product, and $E^2 = \boldsymbol{E} \cdot \boldsymbol{E}$. The integrals in eqn. 1 are over the cavity volume (which includes the detector), and $dV = dx\, dy\, dz$. The proof of eqn. 1 is reported elsewhere [10].

One important feature of this simple formula is that the linewidth predicted for a given power is independent of the distance between the laser and matched detectors of radiation, as one expects on physical grounds. This is because travelling wave fields contribute neither to the integral in the denominator of eqn. 1 provided that the medium outside the laser diode is nondispersive, nor, trivially, to the numerator. The case of dispersive outer media will be discussed at the end of this paper.

A more general form of eqn. 1 is

$$\Delta f P = 4(hf/2\pi)\left\{\int (\boldsymbol{E} \cdot 2\pi f \varepsilon_1 \boldsymbol{E}^* + \boldsymbol{H} \cdot 2\pi f \boldsymbol{\mu}_1 \boldsymbol{H}^*) \, dV\right\}$$
$$\times \left\{\int (\boldsymbol{E}^\dagger \cdot 2\pi f \varepsilon_1 \boldsymbol{E}^{\dagger *} + \boldsymbol{H}^\dagger \cdot 2\pi f \boldsymbol{\mu}_1 \boldsymbol{H}^{\dagger *}) \, dV\right\}$$
$$\times \left| \int (\boldsymbol{E}^\dagger \cdot \boldsymbol{\varepsilon}' \boldsymbol{E} + \boldsymbol{H}^\dagger \cdot \boldsymbol{\mu}' \boldsymbol{H}) \, dV \right|^{-2} \quad (2)$$

where the imaginary parts of the permittivity tensor $\boldsymbol{\varepsilon}$ and permeability tensor $\boldsymbol{\mu}$ are split into $\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2$ and $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$, respectively. We have defined: $\boldsymbol{\varepsilon}' = d(f\boldsymbol{\varepsilon})/df$ and $\boldsymbol{\mu}' = d(f\boldsymbol{\mu})/df$. In eqn. 2, $\boldsymbol{E}^\dagger$ and $\boldsymbol{H}^\dagger$ denote the 'adjoint' electrical and magnetical fields. These are the resonating fields in a cavity identical to the one under discussion except that $\boldsymbol{\varepsilon}$, and $\boldsymbol{\mu}$ are replaced by their transpose, and the (complex) resonant frequency is the same. In the case of a ring-type cavity, the adjoint fields are the fields propagating in the opposite direction with respect to the initial fields in the transposed medium. In the case where the medium is reciprocal ($\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ symmetrical) the transposed medium is identical to the given medium. If, furthermore, the cavity is folded in on itself (as is usual for semiconductor laser diodes), the adjoint fields coincide with the resonating fields $\boldsymbol{E}$ and $\boldsymbol{H}$. Finally, if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are scalar quantities ε and μ, respectively, and the magnetic losses can be neglected, eqn. 2 reduces to eqn. 1. Eqn. 2 can be further generalised to bi-anisotropic media, but this will not be discussed here [10].

In the case of lumped circuits, eqn. 1 can be written

$$\Delta f P = 4(hf/2\pi)\{\textstyle\sum R_k |I_k|^2\}^2 |\textstyle\sum C_k V_k^2 - L_k I_k^2|^{-2} \quad (3)$$

where the sum extends over all resistances R, capacitances C and inductances L of the circuit. R represents only the absorbing part of the circuit. Stimulated emission is expressed by negative resistances that do not appear explicitly in eqn. 3 but appear indirectly as they are supposed to cancel almost exactly the losses due to positive resistances R. I_k denotes the current through R_k or L_k, and V_k the voltage across C_k. Of course, the numerator could be written as $G_k |V_k|^2$, where $G_k = 1/R_k$, expressing the loss electrically rather than magnetically.

In the present paper, these basic relations will be applied to simple, yet interesting, configurations. It is worthwhile to first recall the classical derivation of the Schawlow-Townes (ST) formula

$$\Delta f P = 2\pi h f (\Delta f_c)^2 \quad (4)$$

where Δf is the full half-power laser linewidth, P the output power, and Δf_c the so called 'cold cavity' linewidth, defined as twice the imaginary part of the complex resonant frequency obtained when stimulated emission is suppressed. The proof of eqn. 4 is given in the following section after Yariv's derivation [11].

2 Parallel *LCR* circuit

For the simple circuit model shown in Fig. 1*a*, the ST formula applies for any value of the parameters. The

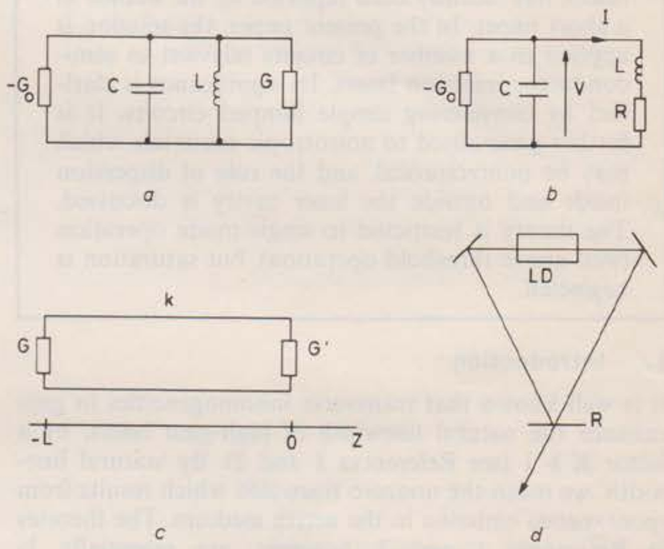

Fig. 1 *Schematic representation of laser oscillators*
a Classical *LRC* parallel circuit
b Series-parallel circuit leading to a departure from the ST formula
c Transmission line equivalent of a conventional laser diode
d Ring-type cavity

active medium is modelled as a constant negative conductance $-G_o (G_o > 0)$ and the cavity as a passive admittance $Y(f) = G + iB(f)$ where the conductance G is a positive constant and the susceptance B is a function of the optical frequency f, in parallel with $-G_o$. At the resonant complex frequency $f_o \equiv f_r + if_i$, we have $Y(f_o) = G_o$, and therefore in the limit where $f_i/f_r \to 0$, $B(f_r) = 0$ and $f_i = (G - G_0)/(dB/df) < 0$. Spontaneous emission is modelled as a current source I_{sp} driving the circuit. Its mean square is

$$|I_{sp}|^2 = 4hfG_o \, df$$

for the spectral range f-$f + df$ and h denotes Planck's constant. Zero temperature is assumed. It is straightforward to show, from eqn. 1, that the spectral density in the load G is

$$S(f) = 4hfG_o G |Y(f) - G_o|^{-2} \quad (5)$$

The full half-power linewidth Δf of the laser is obtained from this expression, assuming a linear variation of B with f in the neighbourhood of f_o. The power P supplied to the load G is obtained by integrating $S(f)$ over frequency. Well above the threshold, the approximation $G_0 \approx G$ can be made in the numerator of eqn. 5. It follows that

$$P \, \Delta f = 8\pi h f G^2 |dB/df|^{-2} \quad (6)$$

If B consists of a parallel LC circuit, we have $B(f) = 1/2\pi fL - 2\pi fC$. Thus, at resonance $dB/df = -4\pi C$ and eqn. 6 can be written

$$P \, \Delta f = (1/2\pi) h f (G/C)^2 \quad (7)$$

which coincides with the ST formula (eqn. 4) $2\pi hf(\Delta f_c)^2$, where $\Delta f_c = -2G/(dB/df)$ denotes the full half-power linewidth of the cold cavity obtained by suppressing G_o.

Alternatively, the general formula (eqn. 3) can be used. Dividing the numerator and the denominator by V^2, we have within the square in the numerator $R(I/V)^2$, where $I/V = 1/R$. In the denominator, we have $C - L(I/V)^2$, where $I/V = (-i2\pi fL)^{-1}$. Note that I in the numerator is the current flowing through the resistance R whereas I in the denominator is the current flowing through the inductance L. They should not be confused. We then find a result identical to eqn. 7.

3 Series parallel *LCR* circuit

This circuit is represented in Fig. 1*b*. The only difference with the circuit treated above is that the resistance R is in series with the inductance L rather than in parallel with it. This, as we shall see, is sufficient to invalidate the ST formula when the quality factor of the circuit is not very high.

The complex resonant frequency f_o of the circuit is given by

$$(G_o + iC2\pi f_o)(R - iL2\pi f_o) = 1 \quad (8)$$

where $-G_o$ represents the active conductance, as before. Thus f_o is given by a 2nd-degree equation. Assuming that the condition

$$4/LC \geqslant (R/L + G_o/C)^2 \quad (9)$$

is fulfilled, the imaginary part f_i of f_o is given by

$$\Delta f = -2f_i = (2\pi)^{-1}(R/L - G_o/C) \quad (10)$$

Well above the threshold, $f_i \to 0$, and therefore $G_o/C \approx R/L$. We can then simplify condition 9 to: $R^2 < L/C$.

The cold cavity linewidth is given by eqn. 10 with G_o suppressed, that is

$$\Delta f_c = (2\pi)^{-1} R/L \quad (11)$$

Application of the ST formula would thus lead to

$$\Delta f P = (2\pi)^{-1} hf (R/L)^2 \quad (12)$$

Application of our general expression (eqn. 3) leads to

$$\Delta f P = 4(hf/2\pi)(R|I|^2)^2 |CV^2 - LI^2|^{-2} \quad (13)$$

where $V/I = R - i2\pi f_o L$. After rearranging, using eqn. 11, we find that $\Delta f P$ is given by the ST formula in eqn. 12 multiplied by the K-like factor

$$K = (1 - R^2 C/L)^{-1} \quad (14)$$

which is close to unity only if $R \ll \sqrt{(L/C)}$. A K factor is thus found even for a rather simple resonant circuit.

4 Laser diode

We consider now a more practical configuration shown in Fig. 1*c*. It consists of a homogeneous active medium from $z = -L$ to $z = 0$. At those locations the power reflectivities are P_R and P'_R, respectively.

Let us model the active homogeneous medium by a uniform transmission line having an impedance per unit length $-R'' - iL''2\pi f$ and an admittance per unit length $-G'' - iC''2\pi f$, where R'' and G'' are positive quantities such that $G''/R'' = C''/L'' = G_c^2$. G_c denotes the (real) characteristic conductance of the transmission line. This line is terminated at $z = -L$ by a conductance $G = G_c(1 - r)/(1 + r)$ and at $z = 0$ by a conductance $G' = G_c(1 - r')/(1 + r')$, where r and r' denote field reflectivities that we consider real for simplicity. The mirror power reflectivities are then $P_R = r^2$ and $P'_R = r'^2$, respectively.

Let $V(z)$ and $I(z)$ denote the voltage and current along the line. To apply eqn. 1 we use the correspondence: $\boldsymbol{E}$, $\boldsymbol{H} \to V$, I; $\sigma \to G' \ \delta(z) + G \ \delta(z + L)$; $\varepsilon' \to C''$, $\mu' \to L''$. Then eqn. 1 becomes

$$\Delta f P = 4(hf/2\pi)\{G_1 |V(-L)|^2 + G'_1 |V(0)|^2\}^2$$
$$\times \left| \int (C'V^2 - L'I^2) \, dz \right|^{-2} \quad (15)$$

where G_1 and G'_1 denote the absorbing parts of G and G', respectively. If the mirrors and loads behind the mirrors are not active, the subscripts '1' in eqn. 15 may be deleted. The integral in eqn. 15 is from $-L$ to 0.

Recall now the classical transmission line formulas

$$\left.\begin{aligned} V(z) &= Z(z) + r'/Z(z) \\ I(z)/G_c &= Z(z) - r'/Z(z) \end{aligned}\right\} \quad (16)$$

where

$$r' = \sqrt{(P'_R)} \quad Z(z) = \exp(ikz) \quad k = (C''2\pi f - iG'')/G_c \quad (17)$$

The resonance condition is

$$Z(L)^2 rr' = 1 \quad (18)$$

and the group velocity $v_g = 2\pi \, df/dk = G_c/C''$.

Introducing eqn. 16 in eqn. 15 leads to the following result:

$$\Delta f P = (hf/2\pi)\tau^{-2}\{(G_1/G)(1 - P_R)/\sqrt{(P_R)}$$
$$+ (G'_1/G')(1 - P'_R)/\sqrt{(P'_R)}\}^2 \quad (19)$$

where $\tau = 2L/v_g$ denotes the laser round trip time.

Let us now consider a few interesting cases. Assume first that the absorbing part is entirely in G'. Then $G_1 = 0$ and $G'_1 = G'$. We obtain the minimum possible linewidth for given τ and P'_R, namely

$$\Delta f P = (hf/2\pi)\tau^{-2}(1 - P'_R)^2/P'_R \quad (20)$$

The result in eqn. 20 is applicable for instance if the transmission line is lossless and gainless, but G is purely active in which case $G_1 = 0$. This result is therefore applicable to a lossless laser diode with unity reflectivity on the left-hand side, coupled on its right-hand side to a long optical fibre terminated by a mirror with reflectivity P'_R. In that configuration, τ represents the fibre round-trip time, if the laser diode round-trip time is neglected. In this so-called 'external cavity-coupled laser' configuration, P'_R is usually small, perhaps 0.001. If there is a nonzero reflection at the right-hand side laser diode facet, that is, between the laser diode and the fiber, the same result again applies, as long as the round-trip time in the laser diode and its losses remain negligible compared to the fibre round-trip time and loss. Note that P in eqn. 20 represents the total dissipated power. The power lost in the fibre, if any, should be subtracted from P to obtain the actual output power. Our result in eqn. 20 essentially agrees with that in Reference 12.

Let us now consider a conventional laser diode with facet power reflectivities P_R and P'_R. Setting $G_1 = G$ and

$G'_1 = G'$ in eqn. 19 we obtain

$$\Delta f P = (hf/2\pi)\tau^{-2}\{(1 - P_R)/\sqrt{(P_R)} + (1 - P'_R)/\sqrt{(P'_R)}\}^2 \tag{21}$$

The factor in braces in eqn. 21 can be written alternatively as

$$(r + r')(g^2 - 1) \tag{22}$$

where $g^2 = 1/rr'$ is the single-pass power gain. This latter expression shows that if the laser gain, the total output power and the diode length are kept fixed, the linewidth Δf is proportional to $(g^2 r + 1/r)^2$, whose minimum value occurs when $r = r' = 1/g$, that is, in a symmetrical configuration. If instead we choose $r = 1$, the linewidth is $(g + 1/g)^2/4 > 1$ times the minimum value.

In a symmetrical configuration: $P_R = P'_R$, and we obtain

$$\Delta f P = 4(hf/2\pi)\tau^{-2}(1 - P_R)^2/P_R \tag{23}$$

This formula should be compared with the result that follows from the ST formula

$$\Delta f P = 4(hf/2\pi)\tau^{-2}(\log(1/P_R))^2 \tag{24}$$

When $P_R = 0.01$, the ST formula in eqn. 24 underestimates the laser linewidth by a factor of about 4.6. The difference between eqns. 23 and 24 has been pointed out before [4, 5].

Note that in the symmetrical configuration being considered, $P = 2P_0$ if P_0 represents the output power per facet. The output powers from the two facets can be recombined with the help of a beam splitter outside the laser because they have a definite relative phase (the nearest axial modes, however, would recombine into the other beam splitter port because they have opposite relative phases).

5 Travelling-wave oscillator

A typical travelling-wave oscillator is shown in Fig. 1*d*. There is a single partially reflecting mirror with power reflectivity P_R, and a laser diode along the closed path free of reflections at the end facets. This of course is an idealised situation. Spontaneous emission excites both clockwise and anticlockwise propagating modes. These two modes have no phase relationship and can be considered independently of each other. In fact, one can always think that there is a nonreciprocal device along the path that pushes frequency in one of these two modes out of the laser diode gain curve, and therefore that only one mode (say, the clockwise mode) is to be considered. Application of eqn. 1 to travelling wave fields would lead to a useless 0/0 result. This is why we must use the more general formulation in eqn. 2. In the present case the 'adjoint' fields are simply counterpropagating fields.

Using, as before, a transmission line analogue with arbitrary but slow variations of $G''(z)$, eqns. 2 can be written

$$\Delta f P = 4(hf/2\pi)\left\{\int (G''|V|^2 + R''|I|^2)\,dz\right\} \times \left\{\int (G''|V^\dagger|^2 + R''|I^\dagger|^2)\,dz\right\} \times \left|\int (C''V^\dagger V + L''I^\dagger I)\,dz\right|^{-2} \tag{25}$$

where, as in Section 4, we have

$$G''/R'' = C''/L'' = G_c^2 \quad k = (C''2\pi f - iG'')/G_c \tag{26}$$

and

$$V(z) = \exp\left\{i\int_0^z k(z)\,dz\right\} \quad I(z) = G_c V(z) \tag{27a}$$

$$V^\dagger(z) = \exp\left\{-i\int_0^z k(z)\,dz\right\} \quad I^\dagger(z) = G_c V^\dagger(z) \tag{27b}$$

The forms in eqn. 27 are applicable to a medium whose parameters vary smoothly with the axial z-co-ordinate. If this were not the case, reflections would take place and we would not be dealing any longer with a ring-type cavity. Note that $V(z)$ and $V^\dagger(z)$ in eqn. 25 can be multiplied by any complex numbers without affecting the result. When the expressions in eqns. 26 and 27 are introduced in eqn. 25 we obtain, without approximation,

$$\int (G''|V|^2 + R''|I|^2)\,dz = \int (G''|V^\dagger|^2 + R''|I^\dagger|^2)\,dz = G_c(\gamma - 1) \tag{28a}$$

$$\int (C''V^\dagger V + L''I^\dagger I)dz = 2C''L'' \tag{28b}$$

where the integrals are over one round trip, γ denotes the round-trip power gain, and L'' the round-trip path length. Therefore

$$\Delta f P = (hf/2\pi)\tau^{-2}(\gamma - 1)^2 \tag{29}$$

In this expression, Δf is the laser full half-power linewidth and P the total dissipated power. If the ring-type cavity does not suffer from any internal loss, but the power is coupled out with the help of a beam splitter of power reflectivity P_R, then P represents the output power and $\gamma = 1/P_R$. Here the parameter τ is again the round trip time L'/v_g. The result in eqn. 29 differs from the result in eqn. 20 which is applicable to a folded cavity, by a factor of P_R instead of $\sqrt{(P_R)}$ in the denominator. Of course, in the limit where $P_R \to 1$, both formulas coincide with the ST formula.

6 Role of the outside medium

It has been indicated in the introduction that the laser linewidth predicted by formulas 1–3 for some value of the output power P is independent of the distance between the laser and some matched detector of radiation, as one expects on physical grounds. Mathematically, this is because the field in that region is in travelling wave form $\exp(ikz)$. For such travelling waves it is well known that

$$\int (\varepsilon E^2 + \mu H^2)\,dx\,dy = 0 \tag{30}$$

the integral being taken here over the cross-section (xy). Therefore, the medium between the laser and the matched detector of radiation makes no contribution to the integral to the denominator of eqn. 1 provided that $\varepsilon' = \varepsilon$ and $\mu' = \mu$, that is, provided that the medium is free of dispersion. This medium makes no contribution to the numerator either provided that it is lossless and gainless. Free space is of course the most common example.

One may be puzzled, however, by the fact that when the medium is dispersive ($\varepsilon' \neq \varepsilon$) but remains lossless and gainless, our formulas predict that the laser linewidth is influenced by the outer medium. Strictly speaking, a dispersive medium must be lossy at some frequencies. However, these frequencies may be outside the frequency