Optical and Quantum Electronics **18** *(1986) 335–343*

Natural linewidth of anisotropic lasers

J. ARNAUD
Equipe de microoptoélectronique de Montpellier, Unité associée au CNRS, USTL, Pl. E. Bataillon 34060 Montpellier Cédex, France

Received 20 March; accepted 22 May 1986

The natural linewidth of lasers is shown to be enhanced with respect to the value predicted by the Schawlow–Townes formula when the gain is high and inhomogeneous in transverse or longitudinal directions. A general formula for the linewidth enhancement is derived from first principles: Maxwell's equations and the fluctuation–dissipation theorem, for media with arbitrary bi-anisotropy and dispersion. The result is expressed in terms of an integral of the resonating field over the cavity volume. For isotropic media (scalar ε and μ), this formula generalizes previous results by Petermann for gain-guided lasers, by Ujihara for lasers with low reflectivity mirrors, and results obtained by other authors for lasers coupled to long external cavities. The role of non-reciprocity is discussed.

1. Introduction

Schawlow and Townes (S–T) have established that the natural (full half-power) linewidth Δf of a well-above-threshold laser is $2\pi(\Delta f_c)^2/P$, where Δf_c is the 'cold' cavity linewidth (defined as Δf, but with the gain suppressed), and P the number of photons emitted per unit time [1]. Saturation effects are omitted, the populations of the upper and lower levels being assumed independent of time.

For most lasers, such as He–Ne lasers, various spurious effects are usually shadowing the natural linewidth Δf. But for semiconductor lasers (LDs for 'laser diodes') the cold cavity linewidth is large, and the natural linewidth can be observed. Measured linewidths of LDs are, nevertheless, significantly larger than the values expected from the S–T formula for two basic reasons: first, time changes in the upper-level population induce changes in the real part of the refractive index. Because this effect is related to saturation it will not be considered further in the present paper. Secondly, in spatially inhomogeneous lasers, the coupling of spontaneous emission to the oscillating mode is enhanced. This is the effect that we will now discuss. In the present paper, previous results are generalized in a number of important ways:

1.1. Transverse effects

Our formula for a transverse linewidth enhancement factor K generalizes and makes more precise Petermann's previous result [2]. Indeed, our formula is applicable to arbitrary anisotropic three-dimensional cavities, whereas Petermann's theory is essentially scalar and one-dimensional. Our integral defining K involves a permittivity factor ε' that was overlooked before. However, this factor can be omitted when spatial changes in group refractive index are small. The K factor is found to be of practical significance for any laser with high transversely inhomogeneous gain, and not specifically for gain guided lasers (i.e. lasers which do not support a true guided mode in the absence of gain). Finally, we directly calculate the laser linewidth Δf. It is not always a straightforward matter to evaluate the laser linewidth from the amount of spontaneous-emission power coupled into the oscillating mode.

0306–8919/86 \$03.00 + .12

1.2. Longitudinal effects

When the single-pass gain of a laser is large (e.g. He–Xe laser at 3.5 μm) and the mirror reflectivities are correspondingly low, the linewidth is enhanced by a kind of longitudinal K-factor which is often overlooked; but see a recent paper by Ujihara [3]. We recover Ujihara's formula from the same simple and general formula discussed above. Furthermore, when a laser is coupled to a long fibre, we may apply to thefibre (loaded at one end by the LD) our formula. Our result is in essential agreement with results obtained by other authors using indirect methods: considering first the LD and, subsequently, introducing the fibre as delayed feedback. This more complicated approach seems to be unnecessary, as long as saturation effects can be neglected. Direct application of the standard S–T formula to the fibre would lead to quite incorrect results. This is probably why the direct approach has not been used before.

1.3. Anisotropy

The preceeding discussion applies to media, whether active or passive, that can be characterized by a scalar complex permittivity. We are motivated to consider more general matrix (or tensor) permittivities for two reasons. One is a matter of principle: the formula that we first derived for scalar permittivities [4] involves the square of the electric field (not the modulus square). It becomes ambiguous for travelling-wave fields in ring-type cavities. In the more general formulation given in the present paper, I use instead the product of the field and of an adjoint field which represents waves propagating in the opposite direction in a medium characterized by the transposed permittivity [5]. The two media are of course the same if the matrix permittivity is symmetrical and, in particular, when it is a scalar. This new formulation removes previous ambiguities. The second reason is of a practical nature: non-reciprocal ring-type cavities seem to be useful to minimize the noise. A degree of non-reciprocity is provided by the Faraday effect, observed for example when light at 1.3 μm propagates in a YIG crystal immersed in a magnetic field. Such a non-reciprocal device may be incorporated in a laser cavity. However, we still have a degeneracy (same resonant frequencies) for counter-propagating waves. A twist of the cavity may help remove all degeneracies. The full bi-anisotropy formalism is useful in such configurations.

In the present paper, we view the laser output as linearly amplified spontaneous emission, and only the well-above-threshold condition is considered in which the field is almost in a single spatial mode. Spontaneous emission is modelled as electrical current sources, according to the fluctuation–dissipation theorem. Although the basic formulae can be found in Landau and Lifshitz [6], a generalization to bi-anisotropic media is needed, which will be presented in Appendix B. The rest of the calculation is based on linear electromagnetism. Surprisingly, I have been unable to find in text-books on electromagnetism the formula for the excitation of a cavity by prescribed current sources with the required degree of generality. This formula, which is new to my knowledge, is derived in Appendix A from a variational principle with respect to first-order changes in the electric field and of the adjoint electric field.

In the main text, I first recall the classical S–T formula and then derive the modified S–T formula, and subsequently discuss a few special cases. The main purpose of this paper, however, is to derive the general formula from first principles. Detailed applications will be reported later.

2. Derivation of the Schawlow–Townes formula

The derivation of the S–T formula follows from a simple circuit model. The active medium is modelled as a constant negative conductance $-G_0 (G_0 > 0)$ and the cavity as a passive admittance $Y(f) = G + iB(f)$ (where the conductance G is a positive constant and the susceptance B is a function of the optical frequency f), in parallel with $-G_0$. At the resonant complex frequency $f_0 \equiv f_r + if_i$, we have $Y(f_0) = G_0$, and therefore in the limit where $f_i/f_r \rightarrow 0$, $B(f_r) = 0$ and $f_i = (G - G_0)/(\mathrm{d}B/\mathrm{d}f) < 0$. Spontaneous emission is modelled as a current source I_{sp} driving the

circuit. Its mean square is

$$|I_{sp}|^2 = 4hfG_0 \, \mathrm{d}f \qquad (1)$$

for the spectral range, $f, f + \mathrm{d}f$. h denotes Planck's constant. Zero temperature is assumed. It is straightforward to show from Equation 1 that the spectral density in the load G is

$$S(f) = 4hfG_0G|Y(f) - G_0|^{-2} \qquad (2)$$

The full half-power linewidth Δf of the laser is obtained from this expression, assuming a linear variation of B with f in the neighborhood of f_0. The number of photons supplied to the load G per unit time, P, is obtained by integrating over frequency the ratio $S(f)/hf$. Well above threshold, the approximation $G_0 \approx G$ can be made in the numerator of Equation 2. It follows that

$$P\Delta f = 8\pi G^2(\mathrm{d}B/\mathrm{d}f)^{-2} \qquad (3)$$

If B consists of a parallel LC circuit, we have: $B(f) = 1/2\pi fL - 2\pi fC$. Thus, at resonance $\mathrm{d}B/\mathrm{d}f = -4\pi C$. The S–T formula quoted earlier follows because the r.h.s. of Equation 3 is equal to $2\pi(\Delta f_c)^2$, where $\Delta f_c = -2G/(\mathrm{d}B/\mathrm{d}f)$ denotes the full half-power linewidth of the cold cavity (obtained by suppressing G_0).

3. General formulation

Let us now present our general formulation. We consider a cavity with perfectly conducting walls which encloses both the active region with complete population invertion (modelling the laser) and the passive region consisting of atoms all in the ground state (modelling the detector or some internal losses). In the following, all volume integrals will be over that cavity volume. The integration element $\mathrm{d}V$ stands for $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. In the cavity, the electromagnetic field obeys Maxwell's equations

$$\mathrm{rot}\, \boldsymbol{E} = \mathrm{i}2\pi f\boldsymbol{B} + \boldsymbol{K} \qquad (4a)$$

$$\mathrm{rot}\, \boldsymbol{H} = -\mathrm{i}2\pi f\boldsymbol{D} + \boldsymbol{J} \qquad (4b)$$

with electrical and magnetic current source terms.

We postulate a linear relationship between $\boldsymbol{D}$, $\boldsymbol{H}$ on the one hand, and $\boldsymbol{E}$, $\boldsymbol{B}$ on the other hand (the so-called 'constitutive equations'), which we write in matrix form

$$\begin{pmatrix} 2\pi \mathrm{i} f\boldsymbol{D} \\ \boldsymbol{H} \end{pmatrix} = M \begin{pmatrix} \boldsymbol{E} \\ 2\pi \mathrm{i} f\boldsymbol{B} \end{pmatrix} \qquad (5)$$

where M is a 6×6 matrix, which is a function of x, y, z if the medium is inhomogeneous, and a function of frequency f (even if the medium is non-dispersive). The M matrix formalism, discussed in [7] and in the appendices, is very convenient even if one is interested only in crystals, in which case M is block diagonal. For reciprocal media M is symmetrical, and for lossless media, $M + \tilde{M}^*$ vanishes (a tilde indicates transposition and a star, complex conjugation).

The driving electrical current $\boldsymbol{J}$ and magnetic current $\boldsymbol{K}$ are now lumped into a single 6-current $\boldsymbol{I}$. Similarly, the electric field $\boldsymbol{E}$ and its rotational are lumped into a 6-field, denoted $\boldsymbol{F}$

$$\boldsymbol{I} = \begin{pmatrix} \boldsymbol{J} \\ 0 \end{pmatrix} + M \begin{pmatrix} 0 \\ \boldsymbol{K} \end{pmatrix}; \quad \boldsymbol{F} = \begin{pmatrix} \boldsymbol{E} \\ \mathrm{rot}\, \boldsymbol{E} \end{pmatrix} \qquad (6)$$

The current $\boldsymbol{I}$ at the (real) frequency f close to a resonant frequency f_m excites a field given by (see Appendix A)

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}_m(\boldsymbol{r}) \frac{\int \tilde{\boldsymbol{F}}_m^{\dagger} \boldsymbol{I}\, \mathrm{d}V}{(f - f_m) \int \tilde{\boldsymbol{F}}_m^{\dagger} (\partial M/\partial f) \boldsymbol{F}_m \, \mathrm{d}V} \qquad (7)$$

We now set the hermitian matrix

$$M + \tilde{M}^* \equiv 2(M_2 - M_1) \tag{8}$$

where M_2 expresses the medium gain (stimulated emission) and M_1 expresses the medium loss (stimulated absorption). Schematically, M_2 represents the laser medium with complete population inversion, whereas M_1 represents the detector which absorbs the radiation or any other lossy medium. The spatial distributions of these two matrices may overlap. Equation 8 generalizes the scalar relation $\varepsilon_i = \varepsilon_1 - \varepsilon_2$, where ε_i denotes the imaginary part of the permittivity.

For a time-harmonic source, the power absorbed in the cavity is given by

$$P = \int \tilde{F}^* M_1 F \, dV \tag{9}$$

where F is defined in Equation 6 and given in Equation 7. (F is to be understood as the r.m.s. value.) However, the current I which models spontaneous emission is not a time-harmonic source but a random function of both space and time. The stochastic average of the right-hand side of Equation 9 divided by hf thus gives us the spectral density $S(f)$ of the number of photons absorbed per unit time in the cavity. From Equation 7 and 9 we obtain

$$S(f) = \frac{(\int \tilde{F}_m^* M_1 F_m \, dV) \iint \tilde{F}_m^\dagger(r_1)\langle I(r_1)\tilde{I}^*(r_2)\rangle F_m^{\dagger *}(r_2)\, dV_1\, dV_2}{hf\,|f - f_m|^2 |\int \tilde{F}_m^\dagger(\partial M/\partial f) F_m \, dV|^2} \tag{10}$$

where $dV_1 = dx_1 dy_1 dz_1$, and $dV_2 = dx_2 dy_2 dz_2$. (The vertical bars denote the modulus and $\langle\ \rangle$ denotes a stochastic average.) The correlation of the 6-current I at two space-points, which appears in Equation 10, is derived from the fluctuation–dissipation theorem as Appendix B. Substituting the result given in Equation B8, where the right-hand side refers only to the active part M_2 (see Equation 8), into Equation 10, we obtain the spectral density of the dissipated photons in the cavity

$$S(f) = \frac{(\int \tilde{F}_m^* M_1 F_m \, dV)(\int \tilde{F}_m^{\dagger *} M_2 F_m^\dagger \, dV)}{|f - f_m|^2 |\int \tilde{F}_m^\dagger(\partial M/\partial f) F_m \, dV|^2} \tag{11}$$

where the fact that M_2 is hermitian has been used. The term $|f - f_m|^{-2}$ in Equation 11 shows that the spectral density is lorentzian. It is therefore easy to evaluate the product of the total number of photons absorbed per unit time and the (full half-power) linewidth $\Delta f = 2f_i$, if we set $f_m \equiv f_r + if_i$

$$P\Delta f = 2\pi S(f_r) f_i^2 = 8\pi \frac{(\int \tilde{F}_m^* M_1 F_m \, dV)(\int \tilde{F}_m^{\dagger *} M_2 F_m^\dagger \, dV)}{|\int \tilde{F}_m^\dagger(\partial M/\partial f) F_m \, dV|^2} \tag{12}$$

Equation 12 is the main result of this paper.

In the case of isotropic media ε and μ are scalar quantities and the adjoint electric field can be taken as equal to the resonating field. The general Equation 12 then reduces to

$$P\Delta f = 8\pi \left\{ \int f\varepsilon_1 |E_m|^2 \, dV \right\}^2 \left| \int (\varepsilon' E_m^2 - \mu' H_m^2) \, dV \right|^{-2} \tag{13}$$

where $\varepsilon' = \partial(f\varepsilon)/\partial f$ and $\mu' \equiv \partial(f\mu)/\partial f = \mu_0$, usually.

It follows from Equation 13 that any matched transmission line linking the active part (laser) to the absorbing part (e.g. the detector) does not affect the linewidth as one expects on physical grounds. Mathematically, this is because dispersionless travelling-wave field regions do not contribute to the integral in the denominator of Equation 13, nor, trivially, to the numerator. It is easy to verify that Equation 13 coincides with the S–T formula given earlier for a single LC-circuit. Let us now consider transverse and longitudinal effects.

4. Transverse effects

The transverse effects are usually expressed by a linewidth enhancement factor K with respect to the

prediction of the S–T formula. Into this S–T formula enters the cold-cavity linewidth Δf_c, which can be defined as twice the imaginary part of the complex resonant frequency, or in terms of the resonating field

$$\Delta f_c = 2 \left\{ \int f\varepsilon_1 |E_c|^2 \, dV \right\} \Big/ \int (\varepsilon' |E_c|^2 + \mu' |H_c|^2) \, dV \tag{14}$$

The numerator in Equation 14 is essentially the dissipated power and the denominator the stored energy. However, because we are dealing presently with high-gain lasers, the cold-cavity fields, E_c, H_c, may differ vastly from the active cavity fields and it is not always permissible to replace the former in Equation 14 by the active cavity fields E_m, H_m. However, if the loss ε_1 and the group permittivity ε' do not vary much spatially, any field can be used in Equation 14, and in particular the active-cavity field.

If this is the case Equation 13 can be written in S–T form

$$P\Delta f = 2\pi(\Delta f_c)^2 K_A \tag{15}$$

where

$$K_A = \left\{ \int (\varepsilon' |E_m|^2 + \mu' |H_m|^2) \, dV \right\}^2 \left| \int (\varepsilon' E_m^2 - \mu' H_m^2) \, dV \right|^{-2} \tag{16}$$

generalizes Petermann's result

$$K = \left\{ \int |E_m|^2 \, dx \right\}^2 \left| \int E_m^2 \, dx \right|^{-2} \tag{17}$$

the integrals extending here over the x-coordinate, transverse to propagation, from $-\infty$ to $+\infty$. These K factors are larger than unity unless the fields have constant phases and the E and H fields are 90° out of phase, in which case they are unity.

5. Longitudinal effects

We now consider the plane-wave model in lasers whose material properties are independent of the transverse x, y, coordinates, but depend only on the longitudinal z coordinate. The end facets of a conventional laser diode with reflectivity less than unity alone introduce the kind of longitudinal inhomogeneity that we are discussing,even if the active material is perfectly uniform along the z axis, from say, $z = 0$ to $z = L$.

The conventional formula for the linewidth of such lasers can be derived either from Δf = average spontaneous emission rate in the mode/2π × number of photons in the mode), or from the S–T formula (with the cold-cavity linewidth defined as twice the imaginary part of the complex resonant frequency). This classical result reads (Equation 26 of [9] with the internal losses neglected $g = \alpha_m = L^{-1}\ln(1/R)$, $\alpha = 0$, $P = 2P_0$, $n_{sp} = 1$, and multiplied by two because we are in the unsaturated regime)

$$\Delta f P = (hf/2\pi)(v_g/L)^2(\ln 1/R)^2 \tag{18}$$

where R is the facet power reflectivity, and v_g the group velocity. Equation 13 leads to a different result, namely

$$\Delta f P = (hf/2\pi)(v_g/L)^2(1 - R)^2/R \tag{19}$$

Clearly Equations 18 and 19 agree when $R \to 1$, but the result in Equation 19 is 1.12 times the result in Equation 18 when $R = 0.3$, and 4.6 times when $R = 0.01$, corresponding to a single-pass gain of 20 dB. Ujihara [3] recently noticed that Equation 19 rather than Equation 18 is the correct formula. My formulation unites in a simple comprehensive expression both Petermann's and Ujihara's results.

Appendix A
Excitation of a cavity by a current

The purpose of this appendix is to derive a general formula for the field excited in a cavity by prescribed electric (J) and magnetic (K) current distributions at frequency f. The cavity is supposed to have perfectly conducting walls S. It may contain both active and passive materials. Its quality Q-factor is supposed to be so high, and the excitation frequency f to be so close to the real part of the resonant frequency f_m of a mode of order m, that it is permissible to assume that the resonating field is in that mode m, and is therefore of the form

$$E(r) = AE_m(r) \tag{A1}$$

where $E_m(r)$ denotes the free resonating field (normalized in arbitrary manner), and A a constant proportional to the source strengths that we wish to evaluate.

We have

$$0 = \text{flux through S of } (E^{\dagger} \times H) = \int \text{div}\,(E^{\dagger} \times H)\,\text{d}V = \int (\tilde{H}\,\text{rot}\,E^{\dagger} - \tilde{E}^{\dagger}\,\text{rot}\,H)\,\text{d}V \tag{A2}$$

for any vector field $E^{\dagger}$ normal to the boundary on S. (Tildes denote matrix transposition, and the integral is over the cavity volume.)

Let us now assume that H is a solution of Maxwell's equations with a source term at frequency f. These equations are

$$\text{rot}\,H = -2\pi i f D + J \tag{A3a}$$

$$\text{rot}\,E = 2\pi i f B + K \tag{A3b}$$

where J is the usual current density, while K is a fictitious magnetic current density often used in electromagnetic theory, for convenience.

We postulate the following linear relationship

$$\begin{pmatrix} 2\pi i f D \\ H \end{pmatrix} = M(f)\begin{pmatrix} E \\ 2\pi i f B \end{pmatrix} \tag{A4}$$

where $M(f)$ is a complex 6×6 matrix function of space and frequency. Most authors assume that M is block-diagonal, but this is not always the case.

Using Equations A3 and A4, the last expression in Equation A2 can be written

$$\int (\tilde{F}^{\dagger} M(f) F - \tilde{F}^{\dagger} I)\,\text{d}V = 0 \tag{A5}$$

where we have defined the 6-vectors

$$F \equiv \begin{pmatrix} E \\ \text{rot}\,E \end{pmatrix} \quad F^{\dagger} \equiv \begin{pmatrix} E^{\dagger} \\ \text{rot}\,E^{\dagger} \end{pmatrix} \quad I \equiv \begin{pmatrix} J \\ 0 \end{pmatrix} + M\begin{pmatrix} 0 \\ K \end{pmatrix} \tag{A6}$$

Now it can be shown that the variation of the integral in Equation A5 is of second order at most for a first-order variation of $E^{\dagger}$, provided E obeys the Maxwell's equation in Equation A3 and A4. Indeed, the Euler–Lagrange equations

$$\partial\Lambda/\partial E_i = \sum_{j=1,2,3} (\partial/\partial x_j)\{\partial\Lambda/\partial(\partial E_i/\partial x_j)\} \tag{A7}$$

for the function

$$\Lambda(E, \partial E) \equiv \tilde{U}\begin{pmatrix} E \\ \text{rot}\,E \end{pmatrix} \tag{A8}$$

where the 6-vector U is any function of r, is

$$(I\ \text{rot})U = 0 \tag{A9}$$

(where I denotes the 3×3 unit matrix and 'rot' is considered a 3×3 antisymmetrical matrix operator) as one can show in component form. Application of this result to the integrand in Equation A5 leads to Maxwell's equations in Equations A3, A4, which can be written for the E-field alone

$$(I\ \text{rot})M\begin{pmatrix} E \\ \text{rot}\,E - K \end{pmatrix} = J \tag{A10}$$

(I rot)(MF − J) = 0

Similarly, one can show that the variation of the integral in Equation A5 is of second order at most for a first-order variation of E, provided $E^{\dagger}$ obeys Maxwell's equations without source terms in a medium characterized by the transpose $\tilde{M}$ of M. All field functions are supposed to be continuous and to satisfy the boundary condition on S.

Because of its variational properties, a first-order variation of the integral in Equation A5 with respect to frequency gives

$$(f - f_m)\int \tilde{F}^{\dagger}(\partial M/\partial f)F\,\text{d}V = \int \tilde{F}^{\dagger} I\,\text{d}V \tag{A11}$$

where f_m is the complex resonant frequency of the source-free field, and the real frequency f of the sources is supposed to be very close to f_m, so that the forced field can be assumed to be of the form in Equation A1.

If we now introduce in Equation A11 the relations

$$F = AF_m; \quad F^{\dagger} = BF_m^{\dagger} \tag{A12}$$

we can evaluate the constant A (the constant B drops out), and the field E excited by the current I is found to be

$$F(r) = F_m(r)\frac{\int \tilde{F}_m^{\dagger} I\,\text{d}V}{(f - f_m)\int \tilde{F}_m^{\dagger}(\partial M/\partial f)F_m\,\text{d}V} \tag{A13}$$

which is the desired result. Note that normalization of the E_m or $E_m^{\dagger}$ resonating fields is unnecessary because multiplicative factors would drop out. This relation implies a lorentzian frequency response, as expected.

If the medium is reciprocal, the M matrix is symmetrical, and the field $F_m^{\dagger}$ may be set equal to the cavity field F_m. However, in a ring-type cavity with travelling-wave fields of the form exp $(i\beta z)$, the integral in the denominator of Equation A13 would then be almost zero. Such travelling-wave solutions can be excited alone, without counter-propagating waves, only by travelling-wave currents, so that the numerator vanishes also. Under such circumstances, Equation A13 is inconvenient. For ring-type cavities, it is therefore preferable to take for the adjoint field $E_m^{\dagger}$ the counter-propagating field, which has essentially an exp $(-i\beta z)$ dependence on the axial coordinate z. This term cancels out the exp $(i\beta z)$ dependence of the E_m field.

For arbitrary M matrices, the two waves propagating in the forward direction and the two waves propagating in the backward direction are non-degenerate, that is, their constants of propagation are neither equal nor opposite. But for any field with resonant frequency f_m in M, there is an adjoint field in $\tilde{M}$ corresponding to the same resonant frequency.

Appendix B
Correlation of the 6-current I

Spontaneous emission from atoms in the upper state (population n_2) can be modelled by an electric current density J and a magnetic current density K. These are random functions of space and time whose statistical properties are given below. The gain constant of the medium, on the other hand, is proportional to the difference $n_2 - n_1$ between the population n_2 in the upper state that provides stimulated emission, and the population n_1 in the lower state that causes stimulating absorption. For a dielectric medium, this gain is expressed by the negative of the imaginary part ε_i of the complex permittivity ε. Therefore we expect the variance of the electric current modelling spontaneous emission to be of the form

$$\langle JJ^* \rangle \propto \{n_2/(n_2 - n_1)\}(-\varepsilon_i) \tag{B1}$$

The ratio of the population density n_2 in the upper state to the population n_1 in the lower state can be written in the form of a Boltzmann law

$$n_2/n_1 = \exp(-hf/kT) \tag{B2}$$

When the temperature T is negative the ratio n_2/n_1 is larger than unity. If we use the notation in Equation B2, Equation B1 reads

$$\langle JJ^* \rangle \propto -\varepsilon_i n_{sp}; \qquad n_{sp} = \{1 - \exp(hf/kT)\}^{-1} \tag{B3}$$

The so-called 'spontaneous emission factor' n_{sp} can be given an alternative form

$$n_{sp} = \tfrac{1}{2} - \coth(hf/2kT)/2 \tag{B4}$$

In their book, Landau and Lifshitz use $n_{sp} - \frac{1}{2}$ instead of n_{sp}. The factor $-\frac{1}{2}$ plays a part only when forces are evaluated (Casimir effect). It corresponds to vacuum fluctuations that we presently ignore.

In the case where only the upper state is populated (complete population inversion), we have: $T \to -0$, and n_{sp} is unity. This is the case that we will consider henceforth, as far as the active medium is concerned.

We are seeking the correlations of the electrical current $\boldsymbol{J}$ and magnetic current $\boldsymbol{K}$ at two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ for bi-anisotropic media. These media are useful to model, for instance, moving media or media with spatial dispersion. The most general linear constitutive relations are here conveniently written

$$\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \varepsilon & \xi \\ \zeta & \mu \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \equiv L \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \tag{B5}$$

where L is a 6×6 matrix, basically different from the M matrix defined in Equation A4 unless it is block diagonal ($\xi = 0$, $\zeta = 0$). In terms of the L matrix

$$M \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} 2\pi i f(\varepsilon - \xi\mu^{-1}\zeta) & \xi\mu^{-1} \\ -\mu^{-1}\zeta & (2\pi i f\mu)^{-1} \end{pmatrix} \tag{B6}$$

Following the line of reasoning in [6], we arrive at the correlation for the 6-current $\boldsymbol{C}$ (r.m.s. values)

$$\langle \boldsymbol{C}(\boldsymbol{r}_1)\tilde{\boldsymbol{C}}(\boldsymbol{r}_2)^* \rangle = -hf4\pi i f(L - \tilde{L}^*)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2); \qquad \boldsymbol{C} \equiv \begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{K} \end{pmatrix} \tag{B7}$$

Unless L is block-diagonal, the $\boldsymbol{J}$ and $\boldsymbol{K}$ fluctuations are correlated.

What we need is the correlation of the 6-current $\boldsymbol{I}$ defined in Equation A6. After lengthy but straightforward calculations we find

$$\langle \boldsymbol{I}_1\tilde{\boldsymbol{I}}_2^* \rangle = -2hf(M + \tilde{M}^*)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{B8}$$

where $\boldsymbol{I}_1 = \boldsymbol{I}(\boldsymbol{r}_1)$ and $\boldsymbol{I}_2 = \boldsymbol{I}(\boldsymbol{r}_2)$, which is a remarkably simple result. This result is in fact to be expected because the positivity or negativity of the hermitian matrix $M + \tilde{M}^*$ expresses the medium loss or gain, respectively.

References

1. A. YARIV, 'Quantum Electronics' (Wiley, New York, 1975).
2. K. PETERMANN, *IEEE J. Quantum Electron.* **QE-15** (1979) 566.
3. K. UJIHARA, *ibid.* **QE-20** (1984) 814.
4. J. ARNAUD, *Electron. Lett.* **22** (1986) 538.
5. *Idem*, *Proc. IEEE* **62** (1974) 1561.
6. L. LANDAU and E. LIFSHITZ, 'Electrodynamics of Continuous Media' (Pergamon, Oxford, 1963) p. 361.
7. J. ARNAUD, 'Beam and Fiber Optics' (Academic, New York, 1976) p. 189.
8. *Idem*, *Electron. Lett.* **21** (1985) 538.
9. H. HENRY, *IEEE J. Quantum Electron.* **QE 18** (1982) 259.
10. F. FAVRE, D. LE GUEN and J. C. SIMON, *ibid.* **QE-18** (1982) 1712.