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Momentum of Photons

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In a letter to this Journal, Greenberg and Greenberg[1] used for the momentum of a photon the expression

$$\mathbf{m} = (\hbar\omega/c^2)\mathbf{u}, \tag{1}$$

where $\hbar\omega$ denotes the photon energy, c the speed of light in free space, and $\mathbf{u}$ the energy velocity in the medium. This was criticized by Ratcliff and Peak,[2] who favor for the photon momentum the expression

$$\mathbf{p} = \hbar\mathbf{k}, \tag{2}$$

where $\mathbf{k}$ denotes the wave vector. For nondispersive media with refractive index n, Eqs. (1) and (2) take respectively the forms: $m = \hbar\omega/cn$ and $p = \hbar\omega n/c$, explicitly considered in Refs. 1 and 2.

We would like to point out that these two momenta, $\mathbf{m}$ and $\mathbf{p}$, have different physical significances that are now well understood. In brief, the true (or mass-carrying) momentum $\mathbf{m}$ enters in conservation laws established in the laboratory frame, while the generalized (or pseudo) momentum $\mathbf{p}$ enters in conservation laws established in the medium frame.[3] In the latter case, the momentum of the medium itself is ignored.

Let us first discuss circumstances where the momentum $\mathbf{m}$ is most useful. Consider an e.m. wave packet with energy W incident on a body in vacuum. Clearly, the momentum $\mathbf{p}_k$ taken up by the body is the difference between the momentum of the wave packet before and after interaction. To evaluate this quantity we only need to know, besides the scattering properties of the body, the expression for the momentum of the wave packet in free space; it is given by Eq. (1) or Eq. (2). We have:

$$m = p = W/c.$$

When a wave packet is transmitted through a slab free of loss and free of Fresnel reflection (e.g., when $\epsilon = \mu = n$, where ϵ denotes the relative permittivity of the medium, μ its relative permeability, and n its refractive index), the momentum taken up by the slab is equal to zero because the e.m. (electromagnetic) momentum is unchanged. The slab therefore remains at rest if it is originally at rest. It is, however, *displaced* in the process by a length Δz. To evaluate this displacement we can state that the center of mass of the total system,[4] slab plus wave packet, is invariant in the coordinate system in which it is initially at rest.

One finds on that basis that the slab displacement Δz is in the forward direction and is given by[5–7]

$$\Delta z = (W/c^2)(n-1)/m_0, \tag{3}$$

where m_0 denotes the mass of the slab per unit length.

This displacement is most easily interpreted by assuming that Eq. (1), with $\mathbf{u}$ representing the energy velocity of the wave in the medium, is applicable during the time the wave packet travels in the slab. For an isotropic nondispersive medium with refractive index n, the magnitude of $\mathbf{u}$ in Eq. (1) is c/n. Thus, as the e.m. wave packet enters into the medium (with $n > 1$), it loses momentum. This loss is balanced by the (forward) mechanical momentum $(W/c)(1-1/n)$ taken up by the medium. When the wave packet leaves the slab, opposite effects take place and the slab goes back to rest. Because of the time lag existing between the light pulse entering and leaving the slab, the slab center of mass is displaced (forward) by the length Δz given in Eq. (3).

When $\epsilon \neq \mu$, multiple reflections take place. Nevertheless, the expression for the force $\mathbf{F}(t)$ exerted by the wave on the medium remains simple. Let us consider a linearly-polarized wave propagating in a multilayer medium under normal incidence (z axis). This force $\mathbf{F}(t)$ is just opposite to the sum of the changes $\Delta_i(\cdots)$ in e.m. momentum flow that take place at the planes of discontinuity, numbered $i = 1, 2, \ldots$. According to Eq. (1), the e.m. momentum flow of a wave is $\mathbf{S}\mathbf{u}/c^2$, if $\mathbf{S}$ denotes the energy flow, for a unit-cross-section area. Thus we have

$$F(t) = -\sum_i \Delta_i(S_+u_+ + S_-u_-)/c^2, \tag{4}$$

where the + and − signs refer to forward and backward waves, $S_+=E_+H_+$, $S_-=E_-H_-$, and $u_+=-u_-=c/n$, $n\equiv(\epsilon\mu)^{1/2}$. From Maxwell's equations, we have

$$E_\pm-\mu_0\mu u_\pm H_\pm=0,$$

$$H_\pm-\epsilon_0\epsilon u_\pm E_\pm=0.$$

Using also the fact that the total electric and magnetic fields $E=E_++E_-$, $H=H_++H_-$ are continuous at an interface, Eq. (4) can be written in the simple form

$$F(t)=-\tfrac{1}{2}\sum_i\left[\mu_0H_i^2\Delta_i(\epsilon^{-1})+\epsilon_0E_i^2\Delta_i(\mu^{-1})\right],\qquad(5)$$

where E_i, H_i denote the total fields at plane i.

For the case of a wave incident from free space on a homogeneous medium (ϵ, μ), Eq. (5) reduces to

$$F(t)=\tfrac{1}{2}c^{-1}(\epsilon+\mu-2)(\epsilon\mu)^{-1/2}E(t)H(t),\qquad(6)$$

where $E(t)$, $H(t)$ denote the fields of the wave transmitted in the medium, as shown before by Shockley[6] and Hauss.[8] If we assume further that $\epsilon=\mu=n$, the mechanical momentum is, from Eq. (6)

$$p_k=\int_{-\infty}^{+\infty}F(t)\,dt=(W/c)(1-1/n),\qquad(7)$$

as we have indicated before. Note that this mechanical momentum is in part in the form of the material being dragged along with the light pulse, and in part associated with a surface force at the interface.[8] We are not concerned in this discussion with this splitting, which depends on electro- and magneto-strictive effects, but only with total forces and momenta.

The physical significance of $\mathbf{m}$ and $\mathbf{p}$ appears most clearly when a light beam is refracted at a plane interface. Györgyi[9] has shown in this Journal that the law of refraction can be viewed as a consequence of the conservation law (laboratory frame)

$$d(\mathbf{m}+\mathbf{p}_k)/dt=0,\qquad(8)$$

where $\mathbf{m}$ is given by Eq. (1), and $\mathbf{p}_k$ is the momentum taken up by the medium. This momentum can be evaluated from Abraham's expression for the body force, which is consistent with Eq. (1). The generalized momentum $\mathbf{p}$, Eq. (2), is nevertheless useful because it allows us to ignore the momentum $\mathbf{p}_k$ of the medium in directions where this medium has translational invariance. In particular, at a plane interface, the tangential component of $\mathbf{p}$ is continuous as postulated by the Descartes–Snell law of refraction. The fact that the tangential component of $\mathbf{p}$ is continuous at a plane interface is therefore not an indication that the medium momentum is equal to zero in that plane, but simply that in this particular formulation the medium momentum is to be ignored.

A similar situation in fact exists for charged particles (with charge e) traversing two plane sheets of current having equal and opposite current densities. This double-sheet creates a discontinuity in the potential vector $\mathbf{A}$, which is otherwise uniform. The laws of mechanics show that the tangential component of the *generalized* momentum $\mathbf{p}=\mathbf{m}+e\mathbf{A}$ is continuous at the interface. The momentum taken up by the sheets, however, is opposite to the change in *true* momentum $\mathbf{m}$ of the particle.

If the medium has translational invariance in all three spatial directions, that is, if it is homogeneous, $\mathbf{p}$ enters in conservation laws of the form

$$d(\mathbf{p}+\mathbf{p}_a)/dt=0,\qquad(9)$$

where $\mathbf{p}_a=m^*\mathbf{v}_a$ denotes the momentum of a particle, with apparent mass m^* and velocity $\mathbf{v}_a$, that may interact with the e.m. wave packet in the medium. Equation (9) is applicable, in particular, to absorbing bodies located in inviscid fluids. The validity of Eq. (9), which can be proved with the help of a Doppler-effect argument,[10] has been verified experimentally by Jones and Richard.[11]

To interpret the phenomenon of radiation pressure on an absorber located in a fluid in the laboratory frame, one needs take into account the mechanical momentum p_k' that accompanies the wave packet and the reaction force on the fluid at the absorber.[10] Note that, in general, $m+p_k'$ does not add up to p (see Ref. 8). The reaction force is therefore essential to understand the balance of momenta. (For a detailed discussion, Eq. (3–2) of Ref. 8 is most useful.) The balance of momenta is illustrated in Ref. 10 for an artificial dielectric, namely a meander line. In that special case, because p_k' happens to be zero, the reaction on the medium is just equal to

$$m-p=(W/c)(1/n-n)<0.$$

Similar considerations are presumably applicable to the interaction of a light wave with a slit of width $\Delta\mathbf{r}$ in a homogeneous polarizable medium. The angular divergence of the beam is most easily obtained from the condition[12] that the phase-shift across the slit be of the order of one, $\Delta\mathbf{k}\cdot\Delta\mathbf{r}\sim1$ or $\Delta\mathbf{p}\cdot\Delta\mathbf{r}\sim\hbar$, but this formulation is not unique.

A rather clear picture thus emerges from recent works dealing with the momentum of photons in polarizable matter. The true momentum is the momentum $\mathbf{m}$ defined in Eq. (1). However, the generalized momentum $\mathbf{p}$, Eq. (2), is convenient to study interactions taking place in homogeneous media if we are not primarily interested in the momentum of the medium itself.

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[12] See, for example, J. A. Arnaud, J. Opt. Soc. Am. **62**, 290 (1972).