Modes of Propagation of Optical Beams in Helical Gas Lenses

Abstract—The modes of propagation of optical beams in helical gas lenses are obtained in closed form, within the approximation of Gauss, with the help of a quasi-geometrical optics method.

The refractive index distribution of an ideal helical gas lens assumes the form [1], [2]

$$n(x_1, x_2) = 1 - (\eta/2)(x_1^2 - x_2^2) \equiv 1 + \tfrac{1}{2}\tilde{\boldsymbol{x}}\,\boldsymbol{\eta}\,\boldsymbol{x},$$

$$\boldsymbol{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \boldsymbol{\eta} \equiv \begin{bmatrix} -\eta & 0 \\ 0 & \eta \end{bmatrix} \tag{1}$$

in an $x_1 x_2$ rectangular coordinate system which rotates about the system axis (z) at a spatial rate $\tau = 2\pi/l$, l being the period of the helices (a positive helicity is assumed). In (1), η denotes a real positive constant proportional to the difference in temperature between the helices. It is the purpose of this letter to give an explicit expression for the modes of propagation in such a medium.

With respect to the rotating coordinate system, the paraxial ray equations relating the position vector $\boldsymbol{q}$ of a ray to its direction vector $\boldsymbol{p}$ are

$$\dot{\boldsymbol{q}} = \boldsymbol{p} + \tau \boldsymbol{q} \tag{2a}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{\eta} \boldsymbol{q} + \tau \boldsymbol{p} \tag{2b}$$

where the upper dots denote differentiation with respect to z, and

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$$\tau \equiv \begin{bmatrix} 0 & \tau \\ -\tau & 0 \end{bmatrix}. \qquad (2c)$$

Equations (2a) and (2b) are straightforward generalizations of the well-known ray equations. Note, incidentally, that if the axis of the optical waveguide were curved, with a radius of curvature ρ and a torsion τ_0, it would be necessary to replace τ in (2) by $\tau+\tau_0$, and subtract from the right-hand side of (2b) a vector of magnitude ρ^{-1} directed along the principal normal. For simplicity, we shall assume that the system axis is a straight line.

A general procedure to obtain the modes of propagation in optical systems lacking meridional planes of symmetry, such as the one presently considered, is described in [3]. This procedure consists of introducing two complex square matrices $\boldsymbol{Q}$ and $\boldsymbol{P}$ which satisfy the ray equations, (2). We have, by definition,

$$\dot{\boldsymbol{Q}} = \boldsymbol{P} + \boldsymbol{\tau Q} \qquad (3a)$$

$$\dot{\boldsymbol{P}} = \boldsymbol{\eta Q} + \boldsymbol{\tau P}. \qquad (3b)$$

Using (3), it is not difficult to show that the scalar field

$$E_{00}(x, z) \equiv \exp(-jkz)|\boldsymbol{Q}|^{-1/2} \cdot \exp\left(-j\frac{k}{2}\tilde{\boldsymbol{x}}\boldsymbol{PQ}^{-1}\boldsymbol{x}\right) \qquad (4)$$

where $k \equiv \omega/c$ denotes the free space propagation constant and $|\boldsymbol{Q}|$, the determinant of $\boldsymbol{Q}$, satisfies the parabolic wave equation [3].

The (Gaussian) pattern of the beam described by (4) is completely defined, in amplitude and phase, by the complex wavefront curvature matrix $\boldsymbol{\mu} \equiv \boldsymbol{PQ}^{-1}$ which, according to (3), obeys the matrix Riccati equation:

$$\boldsymbol{\eta} = \dot{\boldsymbol{\mu}} + \boldsymbol{\mu}^2 - \boldsymbol{\tau\mu} + \boldsymbol{\mu\tau}. \qquad (5)$$

Because we are particularly interested in beam patterns that are independent of z in the rotating coordinate system, we set $\dot{\boldsymbol{\mu}}=0$ in (5), and solve for $\boldsymbol{\mu}$. We have

$$j\boldsymbol{\mu} = \tau \tan v \begin{bmatrix} \cos v - \sin v & j \\ j & \cos v + \sin v \end{bmatrix} \qquad (6)$$

where

$$\sin(2v) \equiv \eta\tau^{-2}$$

as we easily verify by substituting (2c) and (6) in (5). This is, in a simplified form, the result obtained by Marié [2] from a totally different approach. Equation (6) shows that the maximum and minimum widths of the beam are oriented, respectively, along the (focusing) x_1 axis and the (defocusing) x_2 axis, a somewhat unexpected result.

$\boldsymbol{\mu}$ being a constant, (3a) is easily integrated. The matrix $\boldsymbol{Q}$ assumes the form

$$\boldsymbol{Q} = \exp\left[(\boldsymbol{\mu} + \boldsymbol{\tau})z\right] \cdot \boldsymbol{Q}_0 \qquad (7)$$

where $\boldsymbol{Q}_0$ denotes a constant matrix.

According to (4), the on-axis field ($\boldsymbol{x}=\boldsymbol{0}$) is proportional to $|\boldsymbol{Q}|^{-1/2}$. The determinant of $\boldsymbol{Q}$, using the identity of Jacobi [4] and the fact that $\boldsymbol{\tau}$ is traceless, is

$$|\boldsymbol{Q}| = |\boldsymbol{Q}_0| \exp\left[z \operatorname{trace}(\boldsymbol{\mu})\right]. \qquad (8)$$

The propagation constant of the fundamental mode is therefore

$$\beta = k - \operatorname{trace}(j\boldsymbol{\mu}/2) = k - \tau \sin v. \qquad (9)$$

Notice that, because $\boldsymbol{\mu}$ is independent of k, the group velocity $\partial\omega/\partial\beta$ is equal to the velocity of light.

The higher order modes of propagation are obtained [3], [5] by multiplying the fundamental solution, (4), by a Hermite polynomial in two variables

$$\mathrm{He}_{m1m2}(2\boldsymbol{Q}^{*-1}\boldsymbol{x}; \boldsymbol{Q}^{-1}\boldsymbol{Q}^*) \qquad (10a)$$

where the stars denote complex conjugate values, provided $\boldsymbol{Q}_0$ is normalized:

$$\boldsymbol{Q}_0^*\tilde{\boldsymbol{Q}}_0 = 4jk^{-1}(\boldsymbol{\mu}^* - \boldsymbol{\mu})^{-1}. \qquad (10b)$$

Detailed expressions will not be given in this letter. Note, however, that because $\boldsymbol{Q}$ in (10a) is complex, the wavefronts of the various modes do not coincide. This is a characteristic feature of nonorthogonal systems. A notable exception is the case of nonorthogonal resonators with folded optical axes; at the end mirrors of such resonators, $\boldsymbol{Q}$ is real and the wavefronts of all of the modes coincide with the mirror surfaces. Furthermore, the field assumes the same form as in conventional resonators in the oblique coordinate system $\boldsymbol{Q}^{-1}\boldsymbol{x}$. In helical gas lenses, however, the mode patterns are markedly different from those of more conventional optical systems.

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References

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