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Modes in Helical Gas Lenses

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Helical gas lenses incorporate four coaxial helices at temperatures $+T$, $-T$, $+T$, and $-T$, respectively. Because of the resulting change in refractive index of the gas filling the space inside the helices, optical beams can be guided by this system over long distances. A general expression for the modes of propagation is given; it involves Hermite polynomials in two complex variables. For small temperature differences the mode fields reduce to Laguerre-Gauss functions. Calculated irradiance patterns are shown for various mode numbers and various values of T.

I. Introduction

The focusing properties of electrostatic lenses incorporating four coaxial helices at potentials $+V$, $-V$, $+V$, and $-V$, respectively, have been known for a long time in the technology of particle accelerators.[1] These electrostatic lenses evolved from the concept of strong focusing according to which a periodic sequence of converging and diverging lenses with equal absolute powers has a net focusing effect, provided the period does not exceed a certain critical value. Tien and others[2] have discussed the application of this technique to the guidance of optical beams, the four helices being raised at alternately high and low temperatures. Because of the difference in temperature, gradients of refractive index are created in the gas filling the space inside the helices. The gas thus acts as a quadrupole lens whose principal axes rotate along the system axis. Alternatively, the gas can be replaced by a liquid with low optical losses. Refractive index gradients of the type considered can also be induced in electrooptic materials by dc electric fields.

Tien and others[2] gave an approximate expression for the field of the fundamental mode of propagation, applicable when the difference in temperature between the helices is small. A more general expression has been recently obtained by Marié[3] on the basis of the scalar Helmholtz equation. In this paper, we shall use a quasi-geometrical optics approach based on the concept of complex point eikonal.[4] Explicit expressions for modes of arbitrary order are obtained. (Preliminary results were given in Ref. 5.)

In two-dimensional systems the modes of propagation of scalar waves can be represented by the product of a function of Gauss and Hermite polynomials in one real variable, as was first shown by Boyd and Gordon.[6] In the case of systems with rotational symmetry, Laguerre polynomials are to be introduced (Goubau and Schwering[7]). In the more general case of systems such as the helical gas lens that lack meridional planes of symmetry, it is necessary to introduce Hermite polynomials in two complex variables.[8,9] In order to make this paper self-contained, the theory of modes in Gaussian optics is recalled.

II. Modes in Gaussian Optics

As is well known, the eikonal equation of geometrical optics is obtained by substituting in the wave equation field components of the form

$$\psi(\mathbf{r}) = G(\mathbf{r}) \exp[ikS(\mathbf{r})], \qquad (1)$$

where $k = 2\pi/\lambda$ denotes the free-space propagation constant and $\mathbf{r}$ a point in space. Keeping only the terms with the highest power in k, an algebraic relation between the components of ∇S is obtained. In the general case of lossy media this relation, called the eikonal equation, involves complex parameters and the solutions $S(\mathbf{r})$ are complex. Even in lossless media, complex solutions are of interest to account for diffraction effects.

The component $\partial S/\partial z$ of ∇S on the z axis of a x_1x_2z Cartesian coordinate system can be considered a function of $\partial S/\partial x_1$, $\partial S/\partial x_2$, x_1, x_2, and z. We shall make the approximation that $\partial S/\partial z$ is at most of second degree in $\partial S/\partial x_1$, $\partial S/\partial x_2$, x_1, and x_2. This assumption can often be made for paraxial beams provided the eikonal equation is free of singularities in the domain of interest.

Let us consider the case of beams propagating in a direction close to the z axis in isotropic media. The exact eikonal equation is

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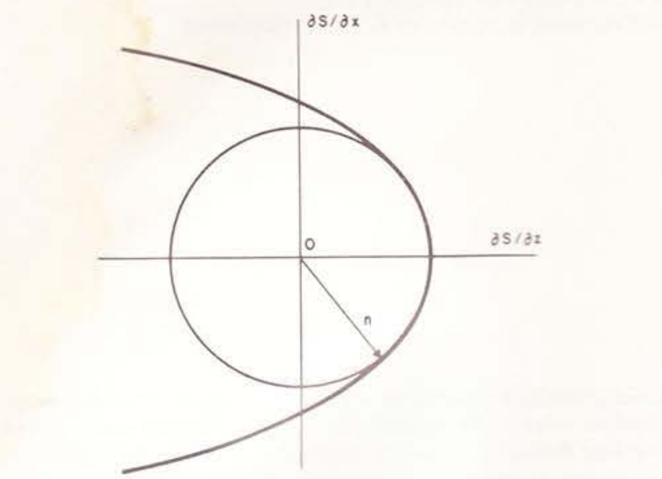

Fig. 1. This figure illustrates the parabolic approximation used for the eikonal equation in the case of beams propagating along the z axis in isotropic media.

$$(\partial S/\partial z)^2 + (\partial S/\partial x_1)^2 + (\partial S/\partial x_2)^2 = n^2(x_1,x_2,z), \quad (2)$$

where the refractive index n can be written approximately

$$n(x_1,x_2,z) \simeq 1 + \tfrac{1}{2}\,\tilde{\mathbf{x}}\mathbf{N}(z)\mathbf{x}, \quad (3)$$

if we assume, for simplicity, that $n(0,0,z)$ is unity and that the medium is aligned. We have introduced a matrix notation in Eq. (3) where column vectors and square matrices are represented by lower- and upper-case boldface letters, respectively, and tildes denote transposition. $\mathbf{x}$ represents a point with coordinates x_1, x_2, and $\mathbf{N}(z)$ is a 2×2 symmetric matrix which characterizes the focusing properties of the medium at some plane z; positive focusing effects, for instance, correspond to negative real definite matrices $\mathbf{N}$.

Equation (2) can be rewritten, within the parabolic approximation (illustrated in Fig. 1)

$$\partial S/\partial z = 1 + \tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{N}\mathbf{x} - \tfrac{1}{2}(\partial S/\partial \mathbf{x})\cdot(\partial S/\partial \mathbf{x}), \quad (4)$$

where $\partial/\partial\mathbf{x}$ denotes the gradient operator in the x_1, x_2 plane.

Let $S(\mathbf{r};\mathbf{r}')$ denote the optical length of a ray going from a point $\mathbf{r}'$ to a point $\mathbf{r}$; S is called the point eikonal. Within the approximation of Gauss this function is, at most, quadratic in $\mathbf{x}$ and $\mathbf{x}'$. Thus, the point eikonal can be written

$$S(\mathbf{r};\mathbf{r}') = d + \tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{U}\mathbf{x} + \tilde{\mathbf{x}}\mathbf{V}\mathbf{x}' + \tfrac{1}{2}\tilde{\mathbf{x}}'\mathbf{W}\mathbf{x}', \quad (5)$$

where $\mathbf{U}$ and $\mathbf{W}$ are 2×2 symmetric matrices and $\mathbf{V}$ is a 2×2 matrix.

d, $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ obey ordinary differential equations that are readily obtained by substituting Eq. (5) in Eq. (4). We get

$$\dot{d} = 1, \quad (6a) \qquad \dot{\mathbf{U}} + \mathbf{U}^2 = \mathbf{N}, \quad (6b)$$

$$\dot{\mathbf{V}} + \mathbf{U}\mathbf{V} = 0, \quad (6c) \qquad \dot{\mathbf{W}} + \tilde{\mathbf{V}}\mathbf{V} = 0, \quad (6d)$$

where the upper dots denote differentiation with respect to z.

The solution of Eq. (6a) is $d = z$. Equation (6b) is a matrix Riccati equation that can be solved in closed form for special functions $\mathbf{N}(z)$ and, in particular, when $\mathbf{N}$ is constant in a fixed or rotating coordinate system.

Let us rewrite Eqs. (6b)–(6d) for the case where the x_1,x_2 coordinate system rotates at a spatial rate τ about the z axis. A rotation τz of the x_1,x_2 coordinate system about the z axis is conveniently expressed in matrix notation by the transformation

$$\mathbf{x} \rightarrow e^{-\mathbf{T}z}\mathbf{x} \quad (7a)$$

where

$$\mathbf{T} \equiv \tau\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (7b)$$

To obtain the transformation of the parameters defining the point eikonal or the refractive index, it suffices to specify that these scalar quantities have the same values in the fixed and rotated coordinate systems. The laws of transformation are therefore obtained by introducing Eq. (7a) in Eqs. (3) and (5). We get

$$\mathbf{N} \rightarrow e^{-\mathbf{T}z}\mathbf{N}e^{\mathbf{T}z}, \quad (8a) \qquad \mathbf{U} \rightarrow e^{-\mathbf{T}z}\mathbf{U}e^{\mathbf{T}z}, \quad (8b)$$

$$\mathbf{V} \rightarrow e^{-\mathbf{T}z}\mathbf{V}, \quad (8c) \qquad \mathbf{W} \rightarrow \mathbf{W}. \quad (8d)$$

Upon substitution of these expressions in Eq. (6), generalized differential equations for $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ are obtained:

$$\dot{\mathbf{U}} - \mathbf{T}\mathbf{U} + \mathbf{U}\mathbf{T} + \mathbf{U}^2 = \mathbf{N}, \quad (9a)$$

$$\dot{\mathbf{V}} - \mathbf{T}\mathbf{V} + \mathbf{U}\mathbf{V} = 0, \quad (9b)$$

$$\dot{\mathbf{W}} + \tilde{\mathbf{V}}\mathbf{V} = 0. \quad (9c)$$

Because these equations are of first order, specification of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ at some plane (say $z = 0$) uniquely defines these matrices everywhere.

To obtain the modes of propagation of optical beams in lenslike media we simply have to expand the Green function in power series of the source coordinates.[10] An asymptotic expression for the field radiated by a point source located at some point $\mathbf{r}'$ in the medium, valid in the limit $k \rightarrow \infty$ (short wavelengths), has been obtained by Van Vleck[11] in connection with quantum-mechanical problems.[12] This expression is

$$\psi(\mathbf{r};\mathbf{r}') = |\partial^2 S(\mathbf{r};\mathbf{r}')/\partial x_i'\partial x_j|^{\frac{1}{2}}\exp[ikS(\mathbf{r};\mathbf{r}')], \quad (10)$$

where the vertical bars denote a determinant. The phase of the square root is defined by continuity from some reference plane. $\psi(\mathbf{r};\mathbf{r}')$ in Eq. (10) is a scalar wavefunction such that $\psi\psi^* ds$ represents the power flowing through a small area ds. In the case of optical beams propagating in a direction close to the z axis in isotropic media, ψ can be taken as $n^{\frac{1}{2}}E$, where n denotes the refractive index and E the magnitude of the electric field, assumed linearly polarized. When Eq. (3) holds, n is almost unity and ψ need not be distinguished from E.

In the case of lossless media and real eikonals, the physical significance of the exponential term in Eq. (10) is straightforward because S represents in that case the optical pathlength from $\mathbf{r}'$ to $\mathbf{r}$ (phase shifts at caustics and focal lines being taken into account). The determinant in front of the exponential, on the other hand, follows from power conservation requirements. Although these interpretations are not applicable to complex eikonals, the expression, Eq. (10), remains a solution of the wave equation, provided the refractive index is analytic for complex $\mathbf{r}$, an assumption that can almost always be made.

It turns out, very fortunately, that Eq. (10) is an exact rather than asymptotic solution of the parabolic wave equation when $S(\mathbf{r};\mathbf{r}')$ is at most quadratic in $\mathbf{x}$ and $\mathbf{x}'$ for every z, z', i.e., within the approximation of Gauss, Eq. (5).[15] Upon substitution of this quadratic form, Eq. (5), in Eq. (10) the field radiated by a point source is found to be

$$\psi(\mathbf{r};\mathbf{r}') = |\mathbf{V}|^{\frac{1}{2}}\exp[ik(z + \tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{U}\mathbf{x} + \tilde{\mathbf{x}}\mathbf{V}\mathbf{x}' + \tfrac{1}{2}\tilde{\mathbf{x}}'\mathbf{W}\mathbf{x}')]. \quad (11)$$

Note that because $\mathbf{V}$ is complex and varies with z, the determinant in front of the exponential contributes to the variation of both the phase and amplitude of the field.

We define now the modes of propagation at some plane z as the coefficients of the expansion of $\psi(\mathbf{r};\mathbf{r}')$ in power series of ikx_1' and ikx_2'. Modes can therefore be viewed as the fields radiated by imaginary multipoles. Explicit expressions are easily obtained if we recall the definition of Hermite polynomials in terms of their generating functions

$$\exp(\tilde{\boldsymbol{\eta}}\boldsymbol{\xi} - \tfrac{1}{2}\tilde{\boldsymbol{\eta}}\boldsymbol{\phi}\boldsymbol{\eta}) = \sum_{m_1,m_2=0}^{\infty}\frac{\eta_1^{m_1}}{m_1!}\frac{\eta_2^{m_2}}{m_2!}He_{m_1m_2}(\boldsymbol{\xi};\boldsymbol{\phi}), \quad (12)$$

where $\boldsymbol{\phi}$ denotes a 2×2 symmetric matrix and $\boldsymbol{\eta},\boldsymbol{\xi}$ denote vectors. Explicitly we have[9]

$$He_{m_1m_2}(\boldsymbol{\xi};\boldsymbol{\phi}) = \sum_{\alpha,\beta,\gamma=0}^{exp.}\frac{(-)^{\gamma}m_1!m_2!\phi_{11}^{\alpha}\phi_{22}^{\beta}\phi_{12}^{\gamma-\alpha-\beta}\xi_1^{m_1-\gamma-\alpha+\beta}\xi_2^{m_2-\gamma-\beta+\alpha}}{2^{(\alpha+\beta)}\alpha!\beta!(\gamma-\alpha-\beta)!(m_1-\gamma-\alpha+\beta)!(m_2-\gamma-\beta+\alpha)!}, \quad (13)$$

where $exp.$ means that the series terminates when one of the exponents becomes equal to zero. It is important to notice that for any diagonal matrix $\mathbf{D}$ with elements λ_1,λ_2 we have

$$He_{m_1m_2}(\mathbf{D}\boldsymbol{\xi};\mathbf{D}\boldsymbol{\phi}\mathbf{D}) = \lambda_1^{m_1}\lambda_2^{m_2}He_{m_1m_2}(\boldsymbol{\xi};\boldsymbol{\phi}), \quad (14)$$

a relation that readily results from the definition Eq. (12) or Eq. (13).

Let us make use of these mathematical results with

$$\boldsymbol{\eta} \equiv ik\mathbf{x}', \qquad \boldsymbol{\xi} \equiv \tilde{\mathbf{V}}\mathbf{x}, \qquad \boldsymbol{\phi} \equiv i\mathbf{W}/k, \quad (15)$$

and expand the right-hand side of Eq. (11) in powers of ikx_1', ikx_2'. The coefficients are, to within unimportant factors,

$$\psi_{m_1m_2}(\mathbf{x},z) = |\mathbf{V}|^{\frac{1}{2}}\exp[ik(z + \tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{U}\mathbf{x})]He_{m_1m_2}(\tilde{\mathbf{V}}\mathbf{x};i\mathbf{W}/k). \quad (16)$$

Equation (16) gives the axial modes of propagation in Gaussian optics in their general form. We note that these fields depend on as many as twenty-two real scalar parameters whose variation with z is defined by Eqs. (6a)–(6d). To be more specific, we may assume that the optical system under consideration occupies the half-space $z \geq 0$, the sources being located at some axial point ($z < 0$) as shown in Fig. 2. The medium on the left of the input plane $z = 0$, called a *mode-generating system*, is chosen such that the desired field configurations are produced at the input plane. Note that the set of modes given in Eq. (16) is biorthogonal to another infinite set denoted $\psi^{\ddagger}_{m_1m_2}(\mathbf{x})$ (see Ref. 9). Arbitrary incident fields $\psi(\mathbf{x})$ can therefore be expanded in series of the modes $\psi_{m_1m_2}(\mathbf{x})$, the coefficients $a_{m_1m_2}$ of the expansion being evaluated by integrating the product of $\psi(\mathbf{x})$ and $\psi^{\ddagger}_{m_1m_2}(\mathbf{x})$ at the input plane.

Let us consider first the fundamental solution, $m_1 = m_2 = 0$. In that case, the Hermite polynomials in Eq. (16) are unity and the field configuration at a plane z depends only on the symmetrical matrix $\mathbf{U}(z)$. The curves of constant irradiance are ellipses if the

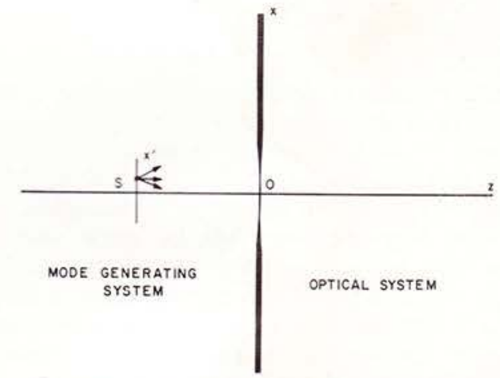

Fig. 2. The optical system under study is shown in the $z \geq 0$ half-space and the mode generating system in the $z < 0$ half-space.

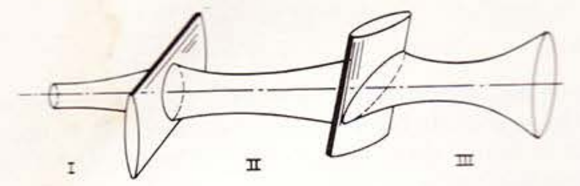

Fig. 3. Transformation of a Gaussian beam with circular symmetry (I), into an astigmatic Gaussian beam (II), and a beam with general astigmatism (III).

imaginary part of $\mathbf{U}$ is positive definite, an assumption that has to be made in order that the mode power be defined. The wavefronts, on the other hand, may have either a positive curvature (converging waves), a negative curvature (diverging waves), or be saddle-shaped, depending on whether the real part of $\mathbf{U}$ is negative definite, positive definite, or indefinite. It should be noted that, in general, the principal axes of the constant phase curves, ellipses, or hyperbolas, do not coincide with those of the constant irradiance curves. These rather unusual mode configurations are observed, for instance, when ordinary Gaussian beams with circular symmetry are launched into a system of cylindrical lenses oriented at some angle different from zero, modulo $\pi/2$, with respect to one another,[17] as illustrated in Fig. 3. The irradiance patterns when m_1 and m_2 are different from zero are rather complicated. In general, the wavefronts do not exhibit simple shapes as in the separable case, and they are different for different mode numbers.

We now apply these general results to helical gas lenses.

III. Modes in Helical Gas Lenses

Helical gas lenses incorporate four coaxial helices at temperatures $+T$, $-T$, $+T$, and $-T$, respectively, as shown in Fig. 4(a). Neglecting aberrations, the refractive index distribution is approximated by a hyperbolic law of the form[2,3,5]

$$n(x_1,x_2) = 1 - \tfrac{1}{2}\eta(x_1^2 - x_2^2) \equiv 1 + \tfrac{1}{2}\tilde{\mathbf{x}}\mathbf{N}\mathbf{x}, \tag{17a}$$

where

$$\mathbf{N} \equiv \begin{bmatrix} -\eta & 0 \\ 0 & \eta \end{bmatrix} \tag{17b}$$

in an x_1,x_2 coordinate system that rotates at a spatial rate $\tau \equiv 2\pi/p$ about the z axis, p being the period of the helices; a positive helicity is assumed. η is proportional to the temperature T of the helices; it is real if the optical losses in the gas are negligible. This assumption, however, need not be made now. For simplicity, we assume that the optical waveguide has a straight axis and that the refractive index is unity on that axis. Because $\mathbf{N}$ is constant in the rotating coordinate system, we shall use the differential equations, Eqs. (9a)–(9c), obeyed by $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ in that system. We are particularly interested in solutions of the field equation that are independent of z except for a factor $\exp(\alpha z)$, where α denotes a constant, because these invariant solutions form a simple basis for the expansion of arbitrary field configurations.

It is clear from Eq. (16) that the field configuration of the fundamental mode ($m_1 = m_2 = 0$) is invariant if the matrix $\mathbf{U}$ does not depend on z. It is easily verified that if we specify that $\dot{\mathbf{U}} = 0$, the solution of Eq. (9a) is[5]

$$\mathbf{U} = \tau \tan\nu \begin{bmatrix} i(\cos\nu - \sin\nu) & 1 \\ 1 & i(\cos\nu + \sin\nu) \end{bmatrix}, \tag{18}$$

where we have defined

$$\sin(2\nu) \equiv \eta\tau^{-2}. \tag{19}$$

Assuming that the medium is lossless, i.e., that η is real, the angle ν defined in Eq. (19) is real when the stability condition

$$\eta < \tau^2 \tag{20}$$

is satisfied. Because we are interested only in beams that carry finite powers, we can restrict ourselves to values of ν included between 0 and $\pi/4$. Equation (18) coincides with an expression recently obtained by Marié[3] using a totally different approach and, in the limit where η is very small compared with τ^2, with an approximate expression given in Ref. 2.

Because $\mathbf{U}$ is a constant we can easily integrate Eq. (9b):

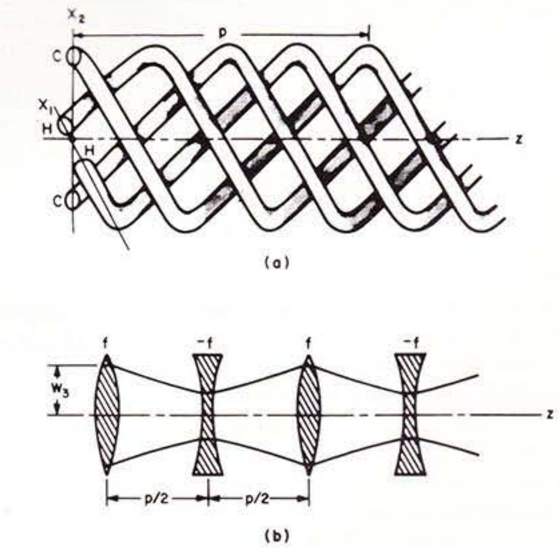

Fig. 4. (a) Helical gas lens (H = hot, C = cold) with helices of period p. (b) View in a meridional plane of symmetry of a sequence of thin quadrupole lenses ($f = 2/\eta p$ and $-2/\eta p$ alternately), whose properties are analogous to those of helical gas lenses.

$$\mathbf{V} = e^{(\mathbf{T}-\mathbf{U})z}\mathbf{V}_0, \tag{21}$$

where $\mathbf{V}_0$ denotes a constant matrix, to be defined later. The matrix $\mathbf{T} - \mathbf{U}$ can be diagonalized using Eqs. (7b) and (18) and written

$$\mathbf{T} - \mathbf{U} = \mathbf{M}\mathbf{D}_0\mathbf{M}^{-1}, \tag{22}$$

where

$$\mathbf{M} \equiv \begin{bmatrix} 1 & i(\cos\nu - \sin\nu) \\ i(\cos\nu + \sin\nu) & 1 \end{bmatrix}, \tag{23}$$

$$\mathbf{D}_0 \equiv \tau \begin{bmatrix} i(\cos\nu - \sin\nu) & 0 \\ 0 & -i(\cos\nu + \sin\nu) \end{bmatrix}. \tag{24}$$

The expression for the $\mathbf{V}$ matrix given in Eq. (21) can consequently be rewritten

$$\mathbf{V} = \mathbf{M}e^{\mathbf{D}_0 z}\mathbf{M}^{-1}\mathbf{V}_0. \tag{25}$$

The factor $|\mathbf{V}|^{\frac{1}{2}}$ in Eq. (16) becomes

$$|\mathbf{V}|^{\frac{1}{2}} = |\mathbf{V}_0|^{\frac{1}{2}} \exp[\tfrac{1}{2}\mathrm{Trace}(\mathbf{D}_0)z]. \tag{26}$$

The propagation constant of the fundamental mode is therefore, from Eqs. (24), (26), and (16),

$$\beta = k - \tau \sin\nu. \tag{27}$$

Let us now look for solutions of Eq. (9c) corresponding to invariant higher order modes. The Hermite polynomial introduced in Eq. (16) is essentially invariant, as we have seen [Eq. (14)], if $\tilde{\mathbf{V}}$ and $\mathbf{W}$ assume the forms

$$\tilde{\mathbf{V}}(z) = \mathbf{D}(z)\tilde{\mathbf{V}}_0, \tag{28}$$

$$\mathbf{W}(z) = \mathbf{D}(z)\mathbf{W}_0\mathbf{D}(z), \tag{29}$$

where $\mathbf{D}(z)$ denotes an arbitrary diagonal matrix and $\mathbf{V}_0$, $\mathbf{W}_0$ constant matrices. $\tilde{\mathbf{V}}(z)$ has the desired form, Eq. (28), if we choose

$$\mathbf{V}_0 = \kappa\mathbf{M}, \tag{30}$$

the factor $\kappa \equiv (k\tau)^{\frac{1}{2}}$ being introduced for later convenience. $\mathbf{V}(z)$ becomes, with that choice for $\mathbf{V}_0$,

$$\tilde{\mathbf{V}}(z) = \kappa\mathbf{D}(z)\tilde{\mathbf{M}}, \tag{31a}$$

where

$$\mathbf{D}(z) \equiv e^{\mathbf{D}_0 z}. \tag{31b}$$

Now substituting Eq. (29), with $\mathbf{D}(z)$ given by Eq. (31b), in the differential equation, Eq. (9c), and using Eq. (31a), we get a linear equation for $\mathbf{W}_0$:

$$\mathbf{D}_0\mathbf{W}_0 + \mathbf{W}_0\mathbf{D}_0 + \tilde{\mathbf{M}}\mathbf{M}\kappa^2 = 0, \tag{32}$$

which has the solution

$$i\mathbf{W}_0/k = \begin{bmatrix} (1/\sin\nu - 1/\cos\nu)^{-1} & i\cot\nu \\ i\cot\nu & (1/\cos\nu + 1/\sin\nu)^{-1} \end{bmatrix}. \tag{33}$$

Introducing these expressions, Eqs. (18), (31), (23), and (24), in the general expression of the field Eq. (16), the complete solution is finally obtained:

$$\begin{aligned} \psi_{m_1m_2}(\mathbf{x},z) &= \exp(ikz) \\ &\times \exp\{i\tau z[(m_1 + 1/2)(\cos\nu - \sin\nu) - (m_2 + 1/2)(\cos\nu + \sin\nu)]\} \\ &\times \exp\{-\tfrac{1}{2}\tan\nu[(\cos\nu - \sin\nu)\chi_1^2 + (\cos\nu + \sin\nu)\chi_2^2]\} \\ &\times \exp[i(\tan\nu)\chi_1\chi_2] \\ &\times He_{m_1m_2}[\chi_1 + i\chi_2(\cos\nu + \sin\nu), \chi_2 + i\chi_1(\cos\nu - \sin\nu); i\mathbf{W}_0/k], \end{aligned} \tag{34a}$$

where

$$\chi \equiv \kappa\mathbf{x} \tag{34b}$$

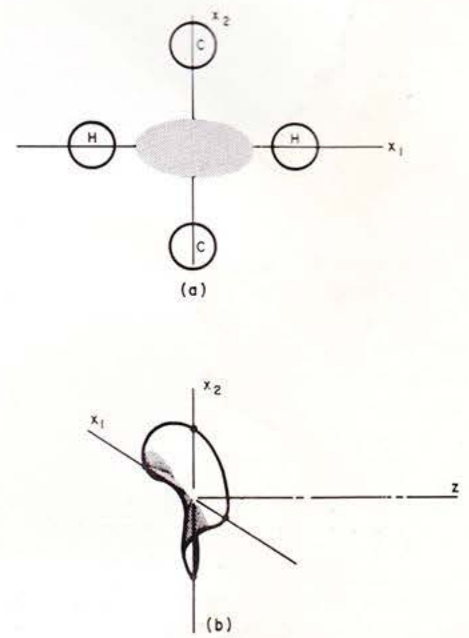

Fig. 5. (a) Irradiance pattern of the fundamental mode. (b) Wavefront of the fundamental mode.

and $i\mathbf{W}_0/k$ is given by Eq. (33). The explicit expression for Hermite polynomials in two variables is given in Eq. (13).

The first exponential term in Eq. (34) corresponds to the geometrical optics phase shift. The second exponential term expresses the phase advance resulting from diffraction. This phase term is of particular importance when the helical gas lens is incorporated in a resonator, because it determines the resonant frequencies. In optical waveguides, knowledge of this phase term is also important because it allows us to compute the transformation of optical beams that are not in a single mode. It should be noted that the group velocity is for all modes, according to this expression, equal to the velocity of light. This is a general consequence of Gaussian optics.

The third exponential term in Eq. (34) is probably the most important from a practical point of view because it defines the size of the beam. The beam contour, corresponding to a field amplitude reduced by a factor $e = 2.718\ldots$ at some transverse plane, is an ellipse whose major and minor semiaxes are, respectively,

$$w_1 = [(k\tau/2)\tan\nu(\cos\nu - \sin\nu)]^{-\frac{1}{2}}, \quad (x_1 \text{ axis}) \qquad (35a)$$

and

$$w_2 = [(k\tau/2)\tan\nu(\cos\nu + \sin\nu)]^{-\frac{1}{2}}, \quad (x_2 \text{ axis}). \qquad (35b)$$

It is interesting to note that the largest beam half-width (w_1) is to be found in the direction of the *focusing*, i.e., the x_1, axis, an unexpected result. The beam irradiance pattern is shown in Fig. 5(a). Equation (35a) can be alternatively written

$$k\eta^{\frac{1}{2}}w_1^2 = 8\pi(\eta p^2)^{-\frac{1}{2}}\cos^2\nu(\cos\nu - \sin\nu)^{-1}. \qquad (36)$$

For comparison, let us consider a sequence of thin quadrupole lenses rotated by 90° every half-period, having the same optical thickness per section as the helical gas lens [i.e., $f = \pm 2/\eta p$, see Fig. 4(b)]. The maximum beam radius w_3 in this system is easily found to be

$$k\eta^{\frac{1}{2}}w_3^2 = 8(\eta p^2)^{-\frac{1}{2}}[(1 + \eta p^2/8)/(1 - \eta p^2/8)]^{\frac{1}{2}}. \qquad (37)$$

The variations of $k\eta^{\frac{1}{2}}w_1^2$ and $k\eta^{\frac{1}{2}}w_3^2$ are shown in Fig. 6 as functions of $\eta^{\frac{1}{2}}p$. This figure shows that the minimum beam sizes w_0 are roughly the same in both cases, though they occur for different periods.

If we assume that the radius of the helices is $\rho = 2w_0$ (a value large enough to ensure negligible interference with the optical beam), a relation is obtained between the total difference in refractive index Δn at adjacent helices and ρ/λ that reads

$$\Delta n \sim 20(\rho/\lambda)^{-2}. \qquad (38)$$

For most liquids ($dn/dT \sim 5 \times 10^{-4}/°C$) and $\rho/\lambda = 100$, this difference in refractive index corresponds to a total temperature difference of about 4°C. In the case of carbon dioxide at a pressure of 10 atmospheres the temperature difference required is about 50°C. The main advantage of confining optical beams by linear refractive index gradients rather than by total reflection, as in cladded fibers, is that group dispersion is eliminated to a large extent.

Let us now consider the higher order modes of propagation. We first note that the irradiance patterns ($\psi\psi^*$) are symmetrical with respect to the x_1 and x_2 axes. The same is not true, however, for the wavefronts. In the limit where $\nu \rightarrow 0$ the diagonal terms of $\mathbf{W}_0$ vanish. Thus, the triple sum in Eq. (13) reduces to a single sum over γ. It is not difficult to show that, in that case, the field of higher order modes is represented by Laguerre-Gauss functions [with an $\exp(il\phi)$ azimuthal dependence]. The irradiance patterns therefore exhibit a circular symmetry for small values of ν. Irradiance patterns were calculated from Eq. (34) for various values of ν, m_1, and m_2. They are shown in Fig. 7 for $m_1 = m_2 = 1$; $m_1 = 2$, $m_2 = 3$; $m_1 = 10$, $m_2 = 1$, and $m_1 = m_2 = 6$. In each case three values of ν—5°, 30°, and 40°—have been considered. As ν increases the beam at first decreases in size and later becomes elongated along the x_1 (focusing) axis.

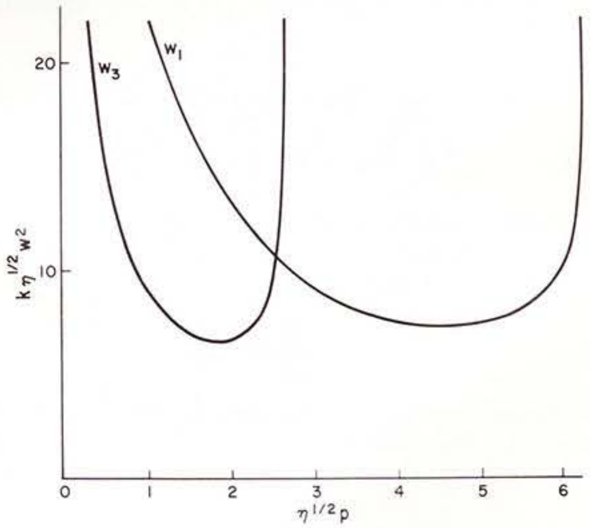

Fig. 6. Comparison between the maximum beam radius in the helical gas lens (w_1) and in a sequence of thin quadrupole lenses (w_3), in reduced coordinates, as a function of the period p. η is proportional to the temperature of the helices and $k \equiv 2\pi/\lambda$.

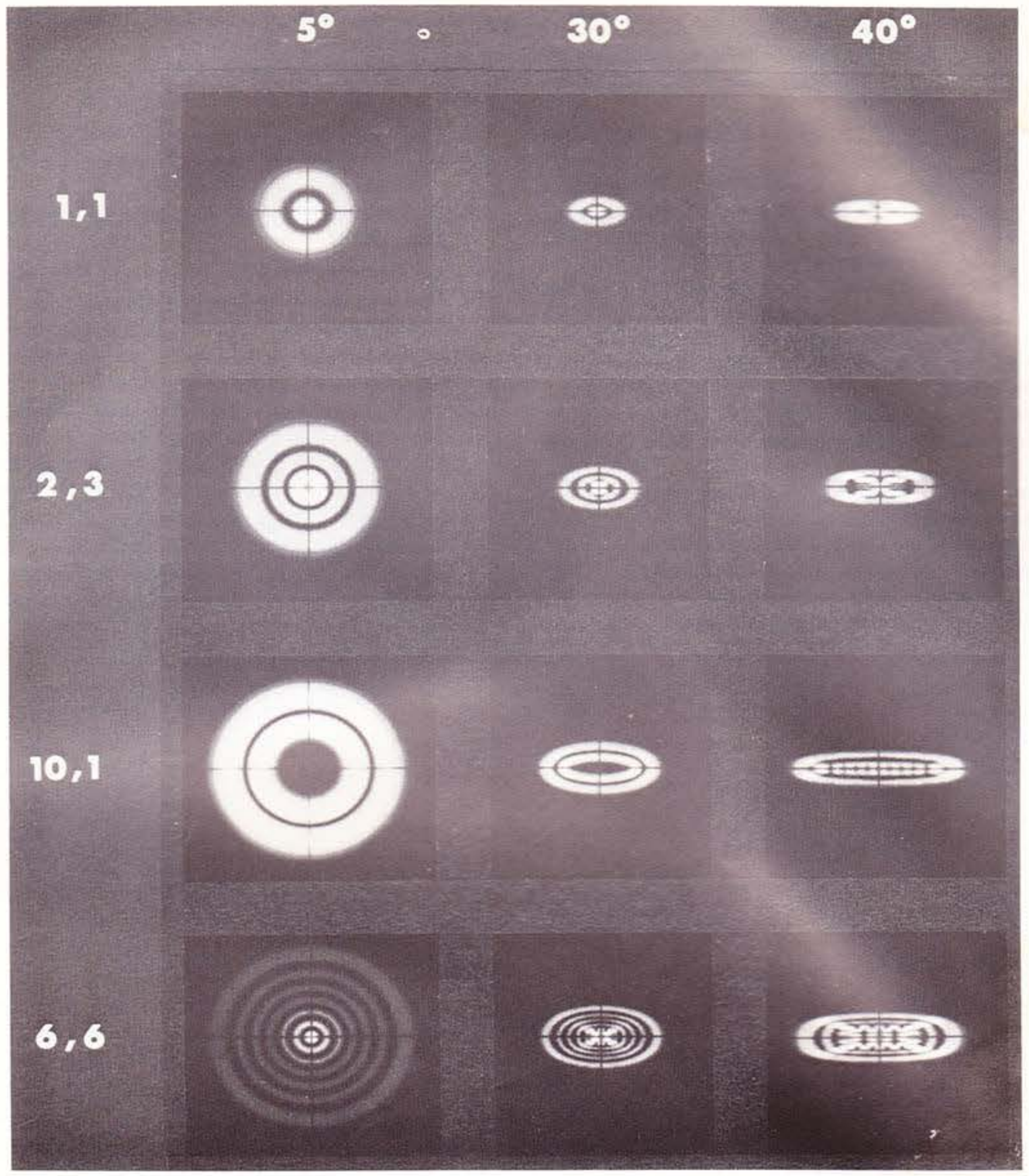

Fig. 7. Mode irradiance patterns in helical gas lenses for various mode numbers and $\nu = 5°$, $\nu = 30°$, $\nu = 40°$. The coordinates are $(k\tau)^{1/2}x_{1,2}$.

These irradiance patterns do not resemble the mode patterns observed in the separable case (see, for instance, Fig. 7 of Ref. 18). The behavior of modes with large mode numbers can, in principle, be described on the basis of Keller's asymptotic theory[19]; this asymptotic theory, however, will not be discussed here.

The main practical difficulty in using helical gas lenses for the guidance of optical beams over long distances is that the beam is eventually intercepted by the walls of the guide if the system suffers from random bends. As is well known, the axes of optical beams propagating in misaligned systems follow classical ray trajectories, provided the approximation of Gauss is applicable.[20] The effect of known misalignments is therefore not difficult to evaluate; this evaluation, however, lies outside the scope of this paper.

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