CHAPTER 2

Periodic Structures

2.1 General Properties of Periodic Structures

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I. Fundamental Considerations about Delay Lines

This section is concerned with the study of circuits used in injection M-type tubes. It is possible to specify the general shape and the characteristic quantities of such circuits; they are periodic in the direction of propagation (of period p and of transverse width l); in most tubes the circuit strutcure can be considered infinite. The most important characteristic of a periodic structure as opposed to a smooth waveguide is its capacity to propagate waves with a delay factor τ greater than unity, τ being the ratio of the light velocity to the phase velocity of the wave. For tubes with output powers varying from a few watts to a number of megawatts, τ takes values between 30 and 4. This is in fact the range of application of the bar line theory which will permit the calculation of many complicated structures.

In M-type tubes, a constant electric field is applied along the interaction space between the line and a smooth plate in front of the line, called the sole. This smooth plate can influence the propagation; however, the product βa of the propagation constant β of the wave and the line–sole distance a is about 1.5 to 5, and it will be shown that this influence can be generally neglected.

The second main parameter of a delay structure is its coupling impedance which denotes the strength of the rf electric field above the line for a given input power P; it is defined by

$$R = \frac{EE^*}{2\beta^2 P}$$

where E is the useful field determined, for instance, at the level of the line, and E^* its conjugate value. R has the same dimension as a resistance and is expressed in ohms.

It is obvious that two identical lines put in parallel, and considered as a whole, have half the coupling impedance of each of them; therefore, the modified coupling impedance given by

$$\mathcal{R} = \frac{Rkl}{\sqrt{\mu_0/\epsilon_0}}$$

where l is the structure width and k, the free propagation constant, is a convenient characterization of the structure field. It varies usually between 0.05 and 1.

The third important parameter is the variation of the phase velocity v_{ph} with the frequency. Let us suppose that the interaction mechanism is such that a variation of τ less than $\Delta\tau$ does not modify the gain by more than 3 db, then the bandwidth Δf will be

$$\frac{\Delta f}{f} = \frac{\Delta\tau/\tau}{(v_{\mathrm{ph}}/v_{\mathrm{g}}) - 1}$$

v_{g} being the group velocity of the electromagnetic wave. One can see that in an amplifier it would be desirable for $v_{\mathrm{ph}}/v_{\mathrm{g}} \simeq 1$, in order to have a wide bandwidth. In fact it is sometimes difficult to make $v_{\mathrm{ph}}/v_{\mathrm{g}}$ less than 2, for reasons which will be analyzed later.

In M-type tubes, the structure must dissipate nearly all the average power distributed *unevenly* along the line. Usually, the thermal conductivity of the metal constituting the line (for copper, the thermal conductivity is about 4 watts per °C per cm) is sufficient to evacuate the heat brought by the beam or due to the rf losses; it is sometimes necessary to use fluid cooling inside the structure itself. There exist two more possibilities of cooling. On the one hand thermal radiation could be conceived

with the use of high melting point metals such as tungsten; besides the difficulty of machining, the radiated power is too small for most tubes. On the other hand, high thermal conductivity insulators such as beryllium oxide could be used to support a structure; this could be advantageous at high delay factors.

It is the purpose of the theoretical developments which will follow to show how to calculate these characteristics. In many cases, such as the usual ladder line, the delay factor cannot be simply predicted with an accuracy better than 10 or 20%. A model must be built (at any scale, however) and it is necessary to know how to relate τ, $\mathcal{R}$, $v_{\mathrm{ph}}/v_{\mathrm{g}}$—the values measured on the model—and how to interpret these results in order to modify the structure in the right way.

The delay structures used in the TWT do not generally involve a lumped circuit and so must be studied by a field theory. However, the delay factor τ being, in general, substantially greater than unity, the electromagnetic waves are tightly bound to the conductors, and it is sometimes possible to assume that the field is stationary at least in some regions or in some particular planes (planes transverse to the bar in the case of bar type lines).

It will be assumed that the structure is perfectly periodic and of infinite length. A direct orthogonal system of reference, $Oxyz$, will be used Oz being the direction of propagation, Ox the direction of the static magnetic field, and Oy the direction of the static electric field. The geometric periodicity p is called the pitch of the line.

The results can be generalized to cylindrical structures. They are established for progressive waves, and can be easily transformed to the case of closed cylindrical structure (magnetron).

II. Floquet's Theorem

The first problem is to find what properties result from the geometrical periodicity for the waves propagating along the structure.

We are interested in solutions of Maxwell's equations with periodic boundary conditions. Let us suppose that we have a complete* set $f_m(z)$ of solutions for some field component for a given structure at a given frequency (the variables x and y have been omitted for conciseness); $f_n(z + p)$ is also a solution because of the geometrical periodicity, and can be expressed by a linear combination of the f_m's:

$$f_n(z + p) = \sum_m \alpha_{nm} f_m(z) \tag{1}$$

* This means only that any solutions at that frequency can be expressed as a linear combination of the functions of the set, not necessarily that the set is complete in the sense of the theory of orthogonal functions.

This relation can be transformed to its diagonal form, and we can define new functions such that

$$F_n(z + p) = \lambda_n F_n(z) \tag{2}$$

The F_n functions are called the modes of the system and two functions F_l and F_m are identical if $F_l(x, y, z) = F_m(x, y, z)$ within a constant factor. It follows that $\lambda_l = \lambda_m$; as usual, let us put $\lambda = e^{-j\varphi}$. Then this last condition can be written:

$$\varphi_l = \varphi_m + 2K\pi \tag{3}$$

K being any integer.

III. Orthonormality Properties of the Modes

In any periodic structure many modes can generally be propagated at a given frequency, the magnitude of each being determined by the input conditions. It is not obvious that the total power flowing through the structure can be calculated by adding the powers of all the modes, calculated separately; the following computation proves that it is true in the case of a loss-less structure. A more general proof is given by Butcher (*1*) who also discusses the use of these properties in traveling wave tube theory and the range of validity of the various theories currently used, which are almost invariably perturbation theories.

We start from the mathematical identity

$$\operatorname{div}[\mathbf{A} \times \mathbf{B}] = \mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B} \tag{4}$$

We first consider two modes 1 and 2 and make

$$\mathbf{A} = \mathbf{E}_1 \qquad \mathbf{B} = \mathbf{H}_2^* \tag{5}$$

and

$$\mathbf{A} = \mathbf{E}_2^* \qquad \mathbf{B} = \mathbf{H}_1 \tag{6}$$

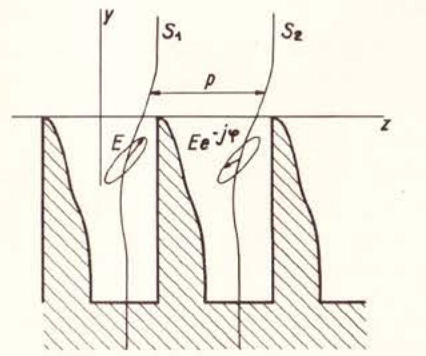

Fig. 1. Uniperiodic structure of pitch p. S_1 and S_2 are any two corresponding surfaces of two adjacent cells. All the components of the field in S_2 differ from those in S_1 by a factor $e^{-j\varphi}$ for a mode.

and add

$$\operatorname{div}[\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1] = \mathbf{H}_2^* \operatorname{curl} \mathbf{E}_1 - \mathbf{E}_1 \operatorname{curl} \mathbf{H}_2^* + \mathbf{H}_1 \operatorname{curl} \mathbf{E}_2^* - \mathbf{E}_2^* \operatorname{curl} \mathbf{H}_1 \tag{7}$$

If we integrate over the volume V of a cell (limiting surfaces S_1 and S_2 shown in Fig. 1), and use Maxwell's equations:

$$\begin{aligned} \operatorname{curl} \mathbf{E} &= -j\omega\mu_0 \mathbf{H} \\ \operatorname{curl} \mathbf{H} &= j\omega\epsilon_0 \mathbf{E} \end{aligned} \tag{8}$$

we obtain

$$\int_{S_1 + S_2} [\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1]\, ds = 0 \tag{9}$$

The integral in S_2 is obviously obtained by multiplying the integral in S_1 by

$$e^{-j\varphi_1} e^{+j\varphi_2^*} \tag{10}$$

Then, unless $\varphi_1 = \varphi_2^*$, we have

$$\int_{S_1} [\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1]\, ds = 0 \tag{11}$$

Now we shall calculate the flow of the Poynting's vector for the total field $\mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{H}_1 + \mathbf{H}_2$:

$$\begin{aligned} P &= \frac{1}{4} \int_{S_1} [(\mathbf{E}_1 + \mathbf{E}_2) \times (\mathbf{H}_1 + \mathbf{H}_2)^* + (\mathbf{E}_1 + \mathbf{E}_2)^* \times (\mathbf{H}_1 + \mathbf{H}_2)]\, ds \\ &= \frac{1}{4} \int_{S_1} (\mathbf{E}_1 \times \mathbf{H}_1^* + \mathbf{E}_1^* \times \mathbf{H}_1)\, ds + \frac{1}{4} \int_{S_1} (\mathbf{E}_2 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_2)\, ds \qquad (12) \\ &+ \frac{1}{4} \int_{S_1} (\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1)\, ds + \frac{1}{4} \int_{S_1} (\mathbf{E}_2 \times \mathbf{H}_1^* + \mathbf{E}_1^* \times \mathbf{H}_2)\, ds \end{aligned}$$

From Eq. (11) it is seen that the last two terms are zero† and

$$P = P_1 + P_2 \tag{13}$$

The condition $\varphi_1 = \varphi_2^*$ means, in the case of propagating waves, that the two modes are in fact the same mode (if we except the case of degenerate modes) and the cross product is no longer zero (in this case $P = P_1 + P_2 + 2\sqrt{P_1 P_2}$).

† This no longer holds when irregularities are present, or near both ends of any structure; in these cases, a coupling exists between the modes and particularly between the useful mode and the TEM mode which can often be propagated between the line and the sole. This problem of "radiation" of guided waves is a very important one when the delay factor is rather small.

IV. Equality of the Magnetic and Electric Energies Stored in a Cell

This classical theorem can be proved from the relation

$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \operatorname{curl} \mathbf{A} - \mathbf{A} \operatorname{curl} \mathbf{B} \tag{14}$$

If we make, $\mathbf{A} = \mathbf{E}$, $\mathbf{B} = \mathbf{H}^*$, and integrate over the cell volume V, we obtain

$$0 = \int_{S_1+S_2} \mathbf{E} \times \mathbf{H}^* \, ds = \int_V - j\omega(\mu_0 \mathbf{H}\mathbf{H}^* - \epsilon_0 \mathbf{E}\mathbf{E}^*) \, dV \tag{15}$$

V. Space Harmonics

It has been seen in Eq. (2) that the periodicity of the structure involves the existence of modes; if $F(x, y, z)$ expresses the dependence of any field component of such a mode, it obeys the relation

$$\frac{F(z+p)}{F(z)} = \lambda_1 \tag{16}$$

λ_1 being a constant number which can alternatively be written $e^{-j\varphi}$. The function $G = F(z)e^{j\varphi z/p}$ is a periodic function of period p and can be developed in a Fourier series:

$$G(z) = \sum_{m=-\infty}^{+\infty} G_m \exp\left(-2m\pi j \frac{z}{p}\right) \tag{17}$$

with

$$G_m = \frac{1}{p} \int_0^p G(z) \exp\left(2m\pi j \frac{z}{p}\right) dz \tag{18}$$

F then has the general form

$$F(z) = \sum_{m=-\infty}^{+\infty} F_m \exp\left[-j(\varphi + 2m\pi) \frac{z}{p}\right] \tag{19}$$

with

$$F_m = \frac{1}{p} \int_0^p F(z) \exp\left[j(\varphi + 2m\pi) \frac{z}{p}\right] dz \tag{20}$$

The field variation along z can be considered as the superposition of "waves" called the space harmonics of propagation constants

$$\beta_m = \frac{\varphi + 2m\pi}{p} \tag{21}$$

with well-defined phase and amplitude relations, depending upon the geometry of the circuit. Space harmonics are of great interest in traveling wave tube theory, because the electron beam usually interacts with a single space harmonic rather than with the full wave.

The related delay factors are given by

$$(c/v_{\mathrm{ph}})_m = \beta_m/k = (\lambda/2\pi p)(\varphi + 2m\pi) \tag{22}$$

For power flowing in the Oz direction, a space harmonic is said to be forward if $\varphi + 2m\pi > 0$, and backward if $\varphi + 2m\pi < 0$.

In general, the magnitude of the space harmonics is computed from some approximate expression of $F(z)$ in a cell; some examples of such computations will be given later (in Section 2.2 on "Theory of Bar Lines" by this author).

Now, let us suppose that the structure is limited by a plane Oxz above which we have free space (for $y > 0$). The total field obeys in free space the wave equation

$$\Delta F + k^2 F = 0 \tag{23}$$

k being the free space propagation constant. With the previous expansion (19) it becomes

$$\sum_{m=-\infty}^{+\infty} \left[\frac{\partial^2 F_m}{\partial x^2} + \frac{\partial^2 F_m}{\partial y^2} + (k^2 - \beta_m{}^2) F_m\right] e^{-j\beta_m z} \tag{24}$$

If we multiply by $e^{j\beta_n z}$ and integrate with respect to z over the pitch, each term of the sum cancels unless $n = m$; this involves the nullity of the mth coefficient,

$$\frac{\partial^2 F_m}{\partial x^2} + \frac{\partial^2 F_m}{\partial y^2} + (k^2 - \beta_m{}^2) F_m = 0 \tag{25}$$

This means that the propagation equation applies to each space harmonic separately. In the special case where F_m does not depend on x (this is the case of the vane type line), F_m decreases exponentially above the structure,

$$F_m(y) = F_m(0) \exp(-\sqrt{\beta_m{}^2 - k^2}\, y) \tag{26}$$

The solution with a sign (+) before the square root is useless here because the field is regular at $y = +\infty$ (in the absence of a sole). One sees that F_m does not decrease above the structure if

$$|\beta_m| \leq k \tag{27}$$

This defines the "forbidden zone" for which the structure can radiate energy.

In general, however, F_m is a function of x and can be expanded in a Fourier integral

$$F_m(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-j\Gamma x} F_{m\Gamma} \, d\Gamma \tag{28}$$

with

$$F_{m\Gamma} = \int_{-\infty}^{+\infty} F_m(x) e^{j\Gamma x} \, dx \tag{29}$$

As previously, we have for each β_m component

$$F_{m\Gamma}(y) = F_{m\Gamma}(0) \exp(-\sqrt{\beta_m{}^2 + \Gamma^2 - k^2}\, y) \tag{30}$$

Usually the field variation along x is slow and we could assume $\Gamma \ll \beta_m$; putting $\beta'_m = \sqrt{\beta_m{}^2 - k^2}$ we have (with a coarse approximation)

$$F_{m\Gamma}(y) \simeq F_{m\Gamma}(0) e^{-\beta'_m y} e^{-\Gamma^2 y/2\beta_m'} \tag{31}$$

It can be shown that $F_m(x, y)$ is deduced from $F_m(x, 0)$ by a Gauss transform

$$F_m(x, y) \simeq e^{-\beta'_m y} \sqrt{\frac{\beta'_m}{2\pi y}} \int_{-\infty}^{+\infty} F_m(X', 0) \exp\left[-\frac{\beta'_m}{2y}(X' - x)^2\right] dX' \tag{32}$$

Physically this means that the sharp variations of the field in the x direction disappear above the structure.

VI. Group Velocity and Energy Velocity

The group velocity in a medium can be considered as the velocity of a pulse shaped signal; in general, such a signal becomes distorted and spreads in space. A velocity can be defined by considering a signal covering a very narrow band of frequencies, or consisting of two waves of slightly different frequencies, $\omega/2\pi$ and $(\omega + \Delta\omega)/2\pi$. In the case of a uniperiodic medium, they have equal phases if

$$\omega t - \beta z = (\omega + \Delta\omega)t - (\beta + \Delta\beta)z \tag{33}$$

The group velocity is defined as

$$v_g = \frac{z}{t} = \frac{\Delta\omega}{\Delta\beta} \to \frac{\partial\omega}{\partial\beta} \tag{34}$$

This definition applied to each space harmonic gives the same value for the group velocity in a periodic system

$$v_g = p \frac{\partial\omega}{\partial\varphi} \tag{35}$$

An energy velocity can also be defined as the ratio of the power flowing through the structure to the average stored energy per unit length

$$v_e = \frac{P}{W} \tag{36}$$

(This definition reminds one of the flow velocity of a liquid in a pipe, which is the ratio of the flow to the quantity of matter per unit length.) It can be proved that $v_g = v_e$; the proof starts as in Section III. Now, in Eq. (7) E_1, H_1 are the fields existing at ω and E_2, H_2 after a little variation $\Delta\omega$ of ω.

At ω, the fundamental phase shift is φ, and at $\omega + \Delta\omega$ it is $\varphi + \Delta\varphi$; then, the integral of the left-hand term of (7) is

$$(e^{j\Delta\varphi^*} - 1)4P \simeq j\Delta\varphi^* 4P \tag{37}$$

and the integral of the right-hand term of (7) is no longer zero because of the variation of ω but $j\Delta\omega 4W$, W being the total energy in a cell. Then, we have (4)

$$\Delta\varphi P = \Delta\omega W \tag{38}$$

or

$$v_g = v_e \tag{39}$$

In the case of a loss-less triperiodic structure the equality between v_g and v_e holds (2). The three periodicities are defined by three vectors $\mathbf{p}_i (i = 1, 2, 3)$ which define a parallelepiped, the face opposited to $\mathbf{p}_i$ being s_i. The propagation in such a structure is defined by the three phase shifts φ_i, or by the propagation constant $\boldsymbol{\beta}$

$$\boldsymbol{\beta} \cdot \mathbf{p}_i = \varphi_i + 2h_i\pi \tag{40}$$

h_i being integers.

The group velocity is defined by

$$\Delta\boldsymbol{\beta} . \mathbf{v}_g = \Delta\omega \tag{41}$$

$\Delta\boldsymbol{\beta}$ being any small vector which satisfies $\Delta\boldsymbol{\beta} = \boldsymbol{\beta}(\omega + \Delta\omega) - \boldsymbol{\beta}(\omega)$. The energy velocity is defined by

$$\mathbf{v}_e = \frac{1}{W} \sum_i P_i \mathbf{p}_i \tag{42}$$

P_i being the power flowing through s_i, and W the stored energy in a cell. The same computation as in the one-dimensional case led to

$$\sum_i \Delta\varphi_i P_i = \Delta\omega W \tag{43}$$

but, from (40) and (42),

$$\Delta\boldsymbol{\beta} \cdot \mathbf{v}_e = \frac{1}{W} \sum_i \Delta\varphi_i P_i = \Delta\omega \tag{44}$$

$\Delta\boldsymbol{\beta}$ having an arbitrary direction. The comparison of (44) and (41) involves $\mathbf{v}_g = \mathbf{v}_e$.

VII. The Reactance Theorem and Dispersion

In some cases, a delay structure can be considered as a filter consisting of pure reactances, generally not lumped. Then, it may be useful to know how such reactances vary with frequency and how their variations act on the dispersion curve of the delay structure.

Let us consider at first a loss-less cavity connected to a coaxial line so that a reactance X can be defined at a given frequency $\omega/2\pi$. If U is the electric energy and T the magnetic energy stored in the cavity, it can be proved (*3*) that the variation of X with ω is given by

$$\frac{\omega}{X}\frac{dX}{d\omega} = \frac{T + U}{T - U} \tag{45}$$

Now, let us suppose, for example, that the equivalent filter is a π with

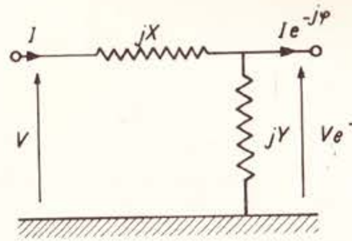

FIG. 2. Filter involving a series reactance X and a shunt suceptance Y, which can be related to a ladder line if X and Y are φ dependent.

shunt suceptance Y and series reactance X (Fig. 2). The classical theory of filters leads to the fundamental phase shift

$$2 \sin\frac{\varphi}{2} = \sqrt{XY} \tag{46}$$

Then, the dispersion is

$$\frac{v_{\mathrm{ph}}}{v_{\mathrm{g}}} = \frac{\tan(\varphi/2)}{\varphi/2}\,\frac{(\omega/X)(\partial X/\partial\omega) + (\omega/Y)(\partial Y/\partial\omega)}{2} \tag{47}$$

from which can be seen that purely capacitive susceptance and purely inductive reactance lead to the least dispersion:

$$\frac{v_{\mathrm{ph}}}{v_{\mathrm{g}}} = \frac{\tan(\varphi/2)}{\varphi/2} \tag{48}$$

In fact, X and Y represent in many delay structures short-circuited or open-circuited bifilar lines of length l, and then

$$\frac{\omega}{X}\frac{\partial X}{\partial\omega} = \frac{\omega}{Y}\frac{\partial Y}{\partial\omega} = \frac{2kl}{\sin 2kl} \tag{49}$$

Furthermore, X and Y depend also on the phase shift because of the coupling between the cells; this dependance on φ usually increases the dispersion. The φ dependance of X can be calculated for some simple structures. For a tape structure, it is found to be

$$\frac{\varphi}{X}\cdot\frac{\partial X}{\partial\varphi} = -\frac{\varphi}{2}\cdot\cot\frac{\varphi}{2} \tag{50}$$

This effect is quite important. In a ladder line, it will double $v_{\mathrm{ph}}/v_{\mathrm{g}}$.

VIII. Perturbation by a Current—General Cases

Let us consider a structure with a fundamental phase shift, φ, at angular frequency, ω, and suppose that a current density, $\mathbf{I}$, at the same angular frequency is introduced in a cell. φ is modified by $\Delta\varphi$ and the problem is to compute how $\Delta\varphi$ depends on $\mathbf{I}$.

We shall use again the expression (7); $\mathbf{E}_1$, $\mathbf{H}_1$ will be now the fields without, and $\mathbf{E}_2$, $\mathbf{H}_2$ the fields with the current. The only change is the fact that the current $\mathbf{I}$ must be introduced in the Maxwell equation for $\mathbf{H}_2$:

$$\operatorname{curl}\mathbf{H}_2 = j\omega\epsilon_0\mathbf{E}_2 + \mathbf{I} \tag{51}$$

The integral of the left-hand term of (7) is

$$j\Delta\varphi 4P \tag{52}$$

the integral of the right-hand term of (*f*) is

$$-\int_v \mathbf{E}_1\mathbf{I}^*\,dv \tag{53}$$

from which

$$\Delta\varphi^* = \left(\frac{j}{4P}\right)\int_v \mathbf{E}_1\mathbf{I}^*\,dv \tag{54}$$

It must be reminded that $\mathbf{E}_1$ is the field before the introduction of the current, and differs (not necessarily slightly) from the total excited field after the current perturbation. Obviously the computation assumes that $\Delta\varphi$ is small compared with φ, but this implies only that $\mathbf{E}_1$ or $\mathbf{I}^*$ or v is small.†

This computation is useful to us in four cases:

A. The Alternating Current of the Beam

The formula is used in the study of the interaction between a beam and a delay structure. In particular, let us consider the case where $\mathbf{I}$ consists of two parallel current sheets of surface current density $\mathbf{I}_s$ and $-\mathbf{I}_s$ distant by e; if we assume that βe is very small before unity, (54) becomes

$$\Delta\beta = -\left(\frac{je\mathbf{I}_s l}{4P}\right)\cdot\frac{\partial\mathbf{E}_z^*}{\partial y} \tag{55}$$

B. The Current Layer Equivalent to Another Delay Structure

The formula can be used to study the coupling between two delay structures of width l. Efficient coupling is provided when one space harmonic of the first structure is nearly synchronous with a space harmonic of the second.

† I is assumed to be zero everywhere in V except in the volume v.

Let us suppose that we have a plane structure, uniform along x, and that the relevant space harmonic has a propagation constant β and a coupling impedance $\mathcal{R}$; the rf field will be obtained from the definition of $\mathcal{R}$.

On the other hand, a layer with a charge density ρ_s corresponds to a current density $\mathbf{I}_s$ such that

$$\operatorname{div}\mathbf{I}_s = -j\omega\rho_s \tag{56}$$

or

$$-j\beta I_{s_z} = -j\omega\rho_s \tag{57}$$

It creates an electric field:

$$E_y = \frac{\rho_s}{2\epsilon_0} \tag{58}$$

Consequently the layer equivalent to the delay structure must have a current density with

$$I_{s_z} = -2j\frac{\omega}{\beta}\epsilon_0 E_z \tag{59}$$

This equivalent current layer directed along the propagation direction must not be confused with the physical current in the structure (for) example, in a helix, where the current does not necessarily flow in the direction of propagation). The second structure can be considered as causing a perturbation of the first, and reciprocally. This gives the propagation constants of the two structures considered as a whole, as a function of their two free propagation constants β_{10} and β_{20}.

$$\beta = \frac{1}{2}(\beta_{10} + \beta_{20}) \pm \left[\left(\frac{\beta_{10} - \beta_{20}}{2}\right)^2 + M^2\right]^{1/2} \tag{60}$$

In the case of two planar structures distant a apart and having coupling impedances $\mathcal{R}_1$ and $\mathcal{R}_2$, the coupling factor M^2 is

$$M^2 = \mathcal{R}_1 e^{-\beta'_{10}a}\mathcal{R}_2 e^{-\beta'_{20}a}\beta_{10}\beta_{20} \tag{61}$$

If the two structures are identical, and $\beta_{10} = \beta_{20} = \beta_0$, $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$

$$\beta - \beta_0 = \pm\beta_0\mathcal{R}e^{-\beta_0'a} \tag{62}$$

C. Perturbation by a Dielectric

The current $\mathbf{I}$ can be a displacement current resulting from the introduction of a material of volume v, with a relative dielectric constant ϵ. Then, we have

$$\mathbf{I} = j\omega\epsilon_0(\epsilon - 1)\mathbf{E}_2 \tag{63}$$

and

$$\Delta\varphi = \left(\frac{\omega\epsilon_0(\epsilon - 1)}{4P}\right)\int_v \mathbf{E}_1^*\mathbf{E}_2\,dv \tag{64}$$

One can limit such a delay structure to n cells so that it can be considered as a cavity with n resonant frequencies for each mode of propagation, the following condition being satisfied:

$$\beta np = K\pi, \qquad K \text{ integer} \tag{65}$$

Then, it is possible to measure the frequency variation Δf resulting from the introduction of the dielectric; Δf can be deduced from the $\Delta\varphi$ of the unlimited structure from the following equation

$$-\Delta f/f = (v_{\mathrm{g}}/v_{\mathrm{ph}})\cdot\Delta\varphi/\varphi \tag{66}$$

From (39), (64), and (66) $\Delta f/f$ can be written

$$\Delta f/f = \frac{1}{2}(1 - \epsilon)\frac{\displaystyle\int_v \mathbf{E}_1\mathbf{E}_2\,dv}{\displaystyle\int_V \mathbf{E}_1\mathbf{E}_2\,dv} \tag{67}$$

where V is the total volume of a cell, and $\mathbf{E}_1$, $\mathbf{E}_2$ are now the fields in the resonant cavity.

Let us consider a special case (Fig. 3); the structure is planar and uniform along x; the perturbating dielectric is a parallelepiped and has the

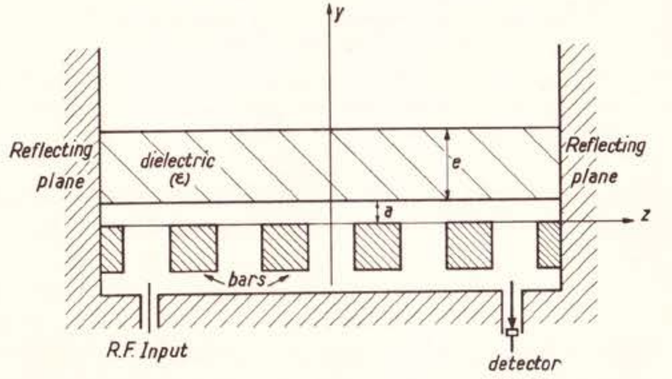

FIG. 3. Cavity formed by a delay structure terminated by two reflecting planes (5 cells). The resonant frequencies are slightly shifted by a piece of dielectric; the coupling impedance of such a structure can be deduced from the frequency shift.

length (along Oz) of the limited structure, its width is l (smaller than that of the structure), its thickness is e and the distance from the structure plane Oxz to the bottom of it is a. Then, using (67) we obtain the value of the coupling impedance from the frequency drift, Δf, by:

$$\mathcal{R} = \frac{v_{\mathrm{ph}}}{v_{\mathrm{g}}}e^{2\beta a}\,\frac{(1 + \epsilon/1 - \epsilon) - (1 - \epsilon/1 + \epsilon)e^{-2\beta e}}{1 - e^{-2\beta e}}\cdot\frac{\Delta f}{f} \tag{68}$$

This expression assumes that one space harmonic only is present in v; this is an important limitation mainly in the case of the interdigital line.

D. The Induced Current in a Real Conductive Layer

In this case $\Delta\varphi$ is purely imaginary as long as the conductivity σ_0 is small, that is to say, as long as the electric field in the vicinity of the layer is only slightly disturbed by the layer; the computation of the atteunation is obvious. When σ is not very small we must take into account an E_y discontinuity across the layer,

$$E_{y_1} - E_{y_2} = \frac{\beta\sigma E_z}{\omega\epsilon_0} \tag{69}$$

In the particular case (Fig. 4) where the conductive layer is rectangular and parallel to Oxz, distant from it by a, of length L, and of width l, we

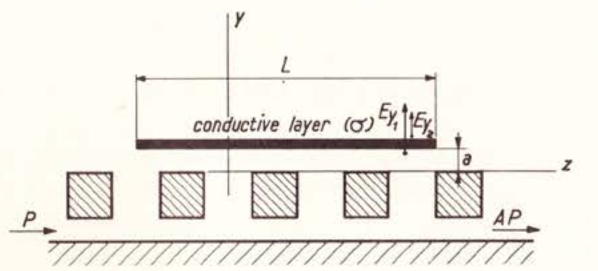

Fig. 4. Delay structure attenuated by a conductive layer of known conductivity; the input power is P, the output power AP; the measurement of A gives the coupling impedance of the structure.

have a power attenuation A (ratio of the output power to the input power for a progressive wave) related to the coupling impedance by

$$\mathcal{R} = -\mathrm{Log}\, A \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{\sigma} \frac{k}{\beta} \frac{e^{2\beta a}}{\beta L}$$

$$\left\{1 + \left[\frac{1}{2}\frac{\beta}{k}\frac{\sigma}{\sqrt{\epsilon_0/\mu_0}}\left(1 - e^{-2\beta a}\right)\right]^2\right\} \tag{70}$$

This expression, like the previous one, is useful in the measurement of the coupling impedance of a structure.

E. The Current Layer Equivalent to Plane Waves Propagating on Two Parallel Biperiodic Structures

In the case of biperiodic structures (54) becomes

$$\Delta\boldsymbol{\beta}\cdot\mathbf{v}_g = \frac{\int_v \mathbf{E}_1^* \mathbf{I}\, dv}{4jW} \tag{71}$$

The same method as in VIII, B led to

$$(\beta - \beta_{10})\frac{\mathbf{v}_{g1}}{v_{g1}} \times (\beta - \beta_{20})\frac{\mathbf{v}_{g2}}{v_{g2}} = M^2 \tag{72}$$

The interaction between the two structures is important only when $\beta_{10} \simeq \beta_{20}$; in these directions $\boldsymbol{\beta}$ can be written, around $\boldsymbol{\beta}_{00} = \boldsymbol{\beta}_{10} = \boldsymbol{\beta}_{20}$,

$$\boldsymbol{\beta} - \boldsymbol{\beta}_{00} = \chi_1\mathbf{u}_1 + \chi_2\mathbf{u}_2 \tag{73}$$

with

$$\chi_1\chi_2 = M^2 \tag{74}$$

and

$$\mathbf{u}_j \frac{\mathbf{v}_{gi}}{v_{gi}} = \delta_{ij} \tag{75}$$

IX. Babinet's Principle

The well-known Babinet principle used in optics can be transposed for the study of tape structures of complementary shape. If a structure is constituted by an infinitely thin conducting sheet with periodic slots, the dispersion curve of a complementary structure (this means that the super-

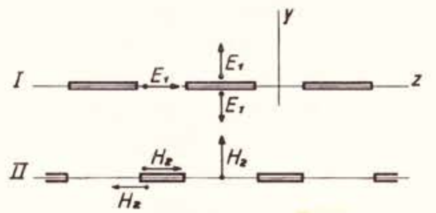

Fig. 5. Complementary tape structures; the magnetic field of structure II is equal, within a constant factor, to the electric field of the structure I.

position of the two constitutes the whole plane, Fig. 5) will be shown to be the same.

It is obvious that in a tape structure, the electric field is in the plane of the structure in a gap and normal to it on the conductor; and reciprocally for the magnetic field. Let us assume that $\mathbf{E}_1$, $\mathbf{H}_1$ are the fields of a wave propagating along structure I; the two sets of fields $\mathbf{E}_2$, $\mathbf{H}_2$ defined by

$$\mathbf{E}_1 \pm \sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{H}_2 = 0$$

$$\mathbf{H}_1 \mp \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}_2 = 0 \tag{76}$$

also satisfy Maxwell's equations. The boundary conditions are satisfied on the structure I by $\mathbf{E}_1$, $\mathbf{H}_1$ and are satisfied on the complementary structure II by $\mathbf{E}_2$, $\mathbf{H}_2$ if we make use of the upper sign above the structure ($y > 0$)

and of the lower sign under the structure ($y < 0$). Then $\mathbf{E}_2$, $\mathbf{H}_2$ constitute the fields of the wave propagating on the structure II. We can remark that

$$\mathbf{E}_1 \times \mathbf{H}_1 = \mathbf{E}_2 \times \mathbf{H}_2 \tag{77}$$

Poynting's vector is the same for the two structures but the electric and magnetic fields are interchanged. As an example, let us consider an infinitely thin interdigital line, and its complementary "zigzag" line; they have the same dispersion curves. In this particular case, in addition, the symmetric space harmonics of the interdigital line become antisymmetric waves in the case of the zigzag line.

List of Symbols

x, y, z	coordinates
$\mathbf{A}, \mathbf{B}$	arbitrary vectors
$\lambda_1, \lambda_1, \lambda_m$	eigenvalues
$\alpha_{n,m}$	numerical coefficients
$f(z), F(z), G(z)$	functions
f	frequency
c	velocity of light
ω	angular frequency
$k\ (= \omega/c)$	propagation constant
$\lambda\ (= c/f)$	free space wavelength
ϵ_0	vacuum permittivity
μ_0	vacuum permeability
v_{ph}	phase velocity
$\boldsymbol{\beta}, \beta\ (= \omega/v_{\mathrm{ph}})$	delayed propagation constants
β_m	propagation constant of the mth space harmonic
$\beta'_m\ (= \sqrt{\beta_m{}^2 - k^2})$	propagation constant in the Oy direction
$\mathbf{p}_i(i = 1, 2, 3), p$	pitch or periodicity of line
$\varphi_i(i = 1, 2, 3), \varphi\ (= \beta p)$	fundamental phase shifts
$\tau\ (= \beta/k)$	delay factor
$v_g\left(= p\dfrac{d\omega}{d\varphi}\right)$	group velocity
l	line width
$P_i(i = 1, 2, 3)$, P	rf power
W	stored energy/unit length or surface
$v_e\ (= P/W)$	energy velocity
$\mathbf{E}, E$	electric field
$R\ (= EE^*/2\beta^2 P)$	Pierce coupling impedance
$\mathcal{R}\ (= Rkl/\sqrt{\mu_0/\epsilon_0})$	coupling impedance
Γ	transverse propagation constant

X	reactance
Y	susceptance
T	stored magnetic energy
U	stored electric energy
$\mathbf{I}$	current density
σ	surface conductivity
ρ_s	surface charge density
a, c	distance in the Oy direction
M	coupling factor
ϵ	relative dielectric constant
L	length of line
A	power attenuation
m, K, n; $h_i(i = 1, 2, 3)$	integers
S_1, S_2, s_i	limiting surface of a cell
V	volume of a cell
v	volume of a dielectric
δ_{ij}	Krocker symbol
$\mathbf{u}_i(i = 1, 2)$	basic vectors
$\mathbf{I}_s, I_s$	surface current density
$\mathbf{H}$	magnetic field
$\mathbf{v}_{gi}$	vector group velocity
$\mathbf{v}_{gi}/v_{gi}$	unit vector directed along vector group velocity
χ_1, χ_2	components of the propagation constant

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