Detuned inhomogeneously broadened laser linewidth

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Abstract. Essentially closed-form formulae are given for the linewidth of inhomogeneously broadened lasers, with arbitrary cavity detuning and pumping statistics (Poissonian or quiet pumps). The so-called 'bad-cavity' mode of operation is treated. The general formula is obtained from a semiclassical theory first derived from a linearized symmetrically ordered quantum theory in the limit of a large number of atoms. This semiclassical theory is most easily explained in terms of independent quantum jumps performed by emitting and absorbing atoms. The effect of inhomogeneous broadening on linewidth is found to be negligible for some pumping schemes, but when the atoms are independently pumped the linewidth is proportional (at some constant power level) to the gain-medium spectral width. We consider, in particular, an intermediate situation in which the rate at which atoms are promoted from the lower to the upper level (pump rate) is proportional to the lower-state population for each atom class. For that particular model, pump fluctuations do not affect the linewidth, even when the cavity is detuned from the centre of the atomic transition frequency distribution.

1. Introduction

In many applications (e.g. optical communication or interferometric detection of gravitational waves) it is important to minimize laser linewidths at some power level. The main purpose of this paper is to show that for some pumping schemes laser linewidths are significantly enhanced by inhomogeneous broadening. This is a consequence of the fact that most atoms are strongly detuned and therefore exhibit large phase–amplitude coupling factors α. It is well known (particularly for semiconductors) that this α-factor may result in linewidth broadening by approximately $1 + \alpha^2$. This crude conclusion, however, needs to be qualified.

When active media are homogeneously broadened (high-pressure gases) the linewidth follows from a simple formula established independently in 1966 by Haken [1] and Lax [2] and quoted in equation (1) later in this paper. This formula shows that the linewidth–power product is constant. It also shows that the linewidth is inversely proportional to the square of the sum of two characteristic times: τ_c, relating to the lossy cavity, and τ_0, relating to the atoms. The so-called 'bad-cavity' mode of operation corresponds to the case $\tau_c \ll \tau_0$. The linewidth is also proportional to a 'spontaneous-emission factor' $n_{sp} \equiv n_e/(n_e - n_a)$, where n_e and n_a denote the fractions of atoms in the upper and lower levels, respectively. For some laser-mirror reflectivities, there is a minimum number, N_0, of active atoms required to permit oscillation at full population inversion. If the laser is constructed with a larger number, N,

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of active atoms, the condition that the gain equals the loss implies that population inversion is incomplete, since the gain, which is proportional to $N(n_e - n_a)$, must remain equal to the mirror loss; we have $N(n_e - n_a) = N_0$. For two-level atoms with $n_e + n_a = 1$, it follows that $n_{sp} = (1 + N/N_0)/2$. But this simple relation does not hold for inhomogeneously broadened lasers.

A medium is called 'inhomogeneously broadened' when it contains atoms whose transition frequencies do not coincide. This may be due to the Doppler effect in gas lasers, to the environment of rare-earth ions in glass-fibre lasers or to the presence of different isotopes. At low temperatures ($T \approx 4$ K), the homogeneous spectral width of glass fibres, for example, falls below the GHz range while inhomogeneous spectral widths remain in the THz range. In this paper only Lorentzian lineshapes are considered and spectral widths (or oscillator linewidths) are defined as full widths at half maximum. The homogeneous spectral width is denoted by $1/\tau_0$ and the spectral width of the atomic transition frequency distribution is denoted by $1/\tau_i$. The medium-gain spectral width is the sum of the two: $1/\tau_e = 1/\tau_0 + 1/\tau_i$. Our second parameter is the ratio $r \equiv \tau_0/\tau_e$ of the medium gain and homogeneous spectral width. We are particularly interested in the limit in which r goes to infinity. For gas lasers, a Gaussian distribution of atomic frequencies is justified by the Maxwell distribution of velocities. But for solid-state lasers the frequency distribution depends on fabrication techniques and Lorentzian shapes are plausible.

Our third parameter, δ, refers to frequency detuning (for brevity, angular frequencies are simply called 'frequencies'). The parameter δ is equal to the difference between the oscillation frequency ω and the atomic-distribution centre frequency ω_i, normalized to the inhomogeneously broadened medium spectral width. We find that the laser linewidth–power product can be expressed in closed form in terms of the three dimensionless parameters just defined, namely $n \equiv N/N_0, r$ and δ. Closed forms will be given, however, only for special cases because the general expression is bulky.

Linewidth formulae applicable to inhomogeneously broadened lasers were given first by Haken [1], Manes and Siegman [3] and recently by Khoury *et al* [4]. Experimental results have been reported by Kuppens *et al* [5]. Our theoretical conclusions depart significantly from those reported in [4]; for example, we find that the linewidth does not depend on pump fluctuations, while, according to [4], it does so significantly. The reason for this discrepancy is that inhomogeneously broadened laser linewidths depend critically on the details of the pumping mechanism. In [4] it is assumed that the pumping rate is the same for all the atoms and is independent of the populations. The pumping scheme considered in the present paper is more realistic, at least for solid-state lasers (e.g. Er^{3+} lasers operating at 1.55 μm and pumped at 1.45 μm). The pumping rate is assumed to be proportional to the lower state population for each atomic class (or spectral packet). As far as the pump fluctuations are concerned, we consider that a single pump, whose fluctuations are characterized by some relative-intensity noise (which vanishes for a Poissonian pump), is shared by the different atoms. The fluctuations of the pump rates for different atoms are correlated, unless the pump is Poissonian. On the other hand, our theory is less general than the one in [4] because some decay rates are neglected, for simplicity.

Our calculations are based on a semiclassical theory, called 'classical (or circuit) theory' to distinguish it from alternative semiclassical theories. Before discussing the specific case of inhomogeneously broadened lasers, let us explain the main concepts, summarizing two recent tutorial papers [6, 7] (see also [8]).

As is well known, quantum theory may be written with either normal or symmetric ordering of the operators. Both methods, if carried out consistently, end up with the same predictions for measurable quantities. But when a quasi-linear approximation is applied

and the operator nature of the optical field is ignored, two distinct semiclassical theories emerge. Normal ordering leads to the so-called 'phasor' theory (much employed in the optical-engineering literature), according to which laser-light fluctuations are caused by the optical power spontaneously emitted in the oscillating mode by excited state atoms.

In 1967, Gordon [9] derived from symmetrically ordered quantum theory a semiclassical theory, which is essentially the one employed in the present paper. In 1988, unaware of that earlier work, I constructed a similar theory on the basis of intuitive arguments [10]. An heuristic explanation of the noise sources is based on the quantum-jump concept introduced by Bohr in 1913. (For a modern discussion relating to quantum jumps, see [11]. The dynamics of isolated atoms discussed in that reference, however, is not important in the present context because we consider a large number of atoms. The continuous measurements responsible for quantum jumps may result from coupling of the atomic states to electronic continua. No consideration of spontaneous emission is required.)

A strictly classical theory provides only the (short-time) average rate at which atoms are raised from the lower to the upper level as a result of stimulated absorption and predicts smooth evolutions of the rates (a similar argument applies to atoms initially in the upper state, absorption being replaced by emission). But assume momentarily that the average rate is constant. Atomic jumps are, from our viewpoint, the fundamental sources of noise. The occurrence times are independent (i.e. Poissonian) because different atoms do not 'communicate' with one another, so to speak: their wavefunctions do not overlap and the optical field has a prescribed value. It follows that one must add to the average jump rate $\langle R \rangle$ a fluctuating rate $r(t)$ whose (double-sided) spectral density is equal to the (short-time) average rate $\langle R \rangle$. This prescription applies to both atoms in the upper and lower states. An application of this principle to detuned atoms shows further that $r(t)$ must be a complex function of time whose real and imaginary parts are uncorrelated and have spectral densities equal to $\langle R \rangle$. This imaginary part of $r(t)$ is responsible for nonzero laser linewidths.

In that theory, it is essential to ensure conservation of particle numbers, as discussed in appendix A of [7]. Pump fluctuations enter at that point. Note that pump fluctuations in no way affect the fundamental $r(t)$ noise sources. They are viewed simply as modulations of the pump power by some random function of time, even in the so-called 'quantum' (or near shot-noise) regime.

According to this picture, laser noise is ascribed to *stimulated* quantum jumps rather than to *spontaneous* emission in the mode. In spite of these vastly different interpretations, the two semiclassical theories just discussed lead to the same expressions for the spectral densities of measurable fluctuations and also agree with quantum theory in the limit of a large number of atoms. There are, however, reasons to favour the classical theory over the phasor theory. One reason is that the phasor theory introduces intermediate random functions of time that are unphysical. For example, the 'light intensity' spectral density (from which, incidentally, the expression 'relative-intensity noise' originates) may be negative. This is meaningless for any measurable function of time. Our classical theory is, on the other hand, much simpler than the exact quantum theory because the large-atom-number approximation is introduced from the very beginning, rather than at the end of difficult calculations.

Let us now go back to the specific problem at hand. For simplicity, a single mode of oscillation is considered. It is assumed that the atoms are submitted to the same relative field-intensity fluctuations. This approximation, also made in [4], applies to lasers provided the mirror reflectivities are close to unity (uniform or mean-field approximation). The difficult case of low-reflectivity mirrors cannot be treated adequately here.

A formula for the linewidth of single-mode lasers containing different pieces of semiconductors, or atoms, was reported in 1988 [10] on the basis of the classical theory

outlined above. This formula accounts for the effect of reduced pump fluctuations and dispersive loads. It is, in principle, applicable to inhomogeneously broadened lasers, except for two assumptions that were made for simplicity, but are not essential.

It is assumed in [10] that the pump rate absorbed by each element is independent of the population difference of that element. This assumption is legitimate for laser diodes because each element is pumped by a high-impedance electrical-current source. In that case the $1+\alpha^2$ factor applies in full and the linewidth may be further enhanced by pump-intensity fluctuations. This assumption would also apply to solid-state lasers if inhomogeneous broadening were caused by independently pumped isotopes. But for most gas or solid-state lasers, different atoms (or ions) share the same pump power and the pump rate absorbed by each atom (or more accurately by each spectral packet) depends significantly on its population. When $N \approx N_0$ the $1+\alpha^2$ factor turns out to be suppressed. But when $N \gg N_0$ inhomogeneous broadening contributes a factor approximately equal to $\frac{5}{8}n$ ($n \equiv N/N_0$). In the large-n limit, the linewidth-enhancement factor tends to the inhomogeneous-to-homogeneous spectral widths ratio r, as reported in [7] without, regrettably, appropriate qualifications.

In [10] load dispersion was accounted for but atomic dispersion was neglected due to the broad gain linewidth of semiconductors. In the present paper, an arbitrary number of atoms, of degree of inhomogeneous broadening and detuning, are considered.

Comparison between the power–linewidth product with and without inhomogeneous broadening is presented in figure 1, for $n = 10$ and $r = 10$, as a function of detuning δ. In the 'good-cavity' case considered in this figure, it is equivalent to increasing r by increasing inhomogeneous broadening or by decreasing the homogeneous spectral width.
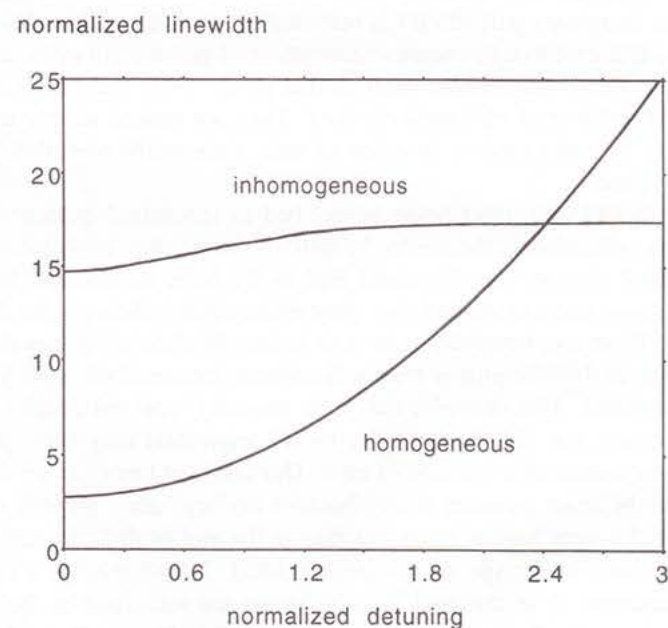

Figure 1. The normalized linewidth $\tau_c^2(P/\hbar\omega)\delta\omega$ in the good-cavity limit ($\tau_c \gg \tau_0$) is plotted as a function of the normalized detuning $\delta \equiv 2\tau_c(\omega - \omega_i)$ from the centre of the atomic-frequency distribution, for a homogeneously broadened laser ($r = 1$) and for an inhomogeneously broadened laser with $r \equiv \tau_0/\tau_c = 10$. The atom-number parameter $n \equiv N/N_0 = 10$.

Detuning turns out to be rather unimportant, as one expects, since most atoms are detuned anyway from the oscillation frequency in strongly inhomogenously broadened lasers. As mentioned earlier in this introduction, much above threshold pump-intensity noise does not affect linewidths.

The results presently reported assume steady-state stable operation. It has been established that inhomogeneously broadened lasers are prone to instabilities. The reader is referred for that problem, not treated here, to the review by Abraham *et al* [12].

In order to verify the present theory experimentally, it would be necessary to construct a laser with strong inhomogeneous broadening and a large number of atoms (ions). Glass-fibre lasers operating at low temperatures ($T \approx 4$ K) would meet these conditions, but it is only recently that such lasers were made to operate single mode and only upper bounds to the linewidth (≈ 1 kHz) were reported, while the quantum-limited linewidths are of the order of 1 Hz.

We find it convenient to employ a circuit representation for both the optical resonator and the atoms, similar to the one presented by Gordon [9]. The resonator is represented by a parallel inductance–capacitance circuit resonating at frequency ω_c. The voltage V across this circuit is proportional to the internal optical field (denoted by A in [4]). The relationship between this electrical circuit and two-mirror resonators commonly employed is discussed, for example, in [6]. The proportionality constants need not be given here because they do not enter in the final formulae. This resonator is supposed to contain absorbing atoms (modelling mirror losses, for example) and emitting atoms differing from each other only by their transition frequencies ω_k, where k labels the atoms.

Each atom is represented by a resonating series inductance–capacitance–resistance circuit resonating at the atomic transition frequency ω_k. The resistance, representing atomic-polarization damping, is inversely proportional to the so-called 'transverse' decay time. In this circuit model, the electric current $I(\omega, d)$ is proportional to the time derivative of the atomic polarization (denoted by M in [4]). Each atom is therefore represented by an admittance $Y(\omega, d) \equiv I(\omega, d)/V$, depending on frequency ω and proportional to the population difference $d \equiv n_e - n_a$.

The fundamental noise source discussed earlier may be represented by a white Gaussian current source $c(t)$ in parallel with the resistance of the atom equivalent circuit. But in the steady-state regime considered, it is simpler to represent noise by complex fluctuating rates $r(t) \equiv r'(t) + ir''(t)$ as discussed earlier. Let us clarify our notation further: an admittance is denoted by $Y \equiv G + iB$; V is understood to be a root-mean-square value and is divided by $\sqrt{\hbar\omega}$, so that the real photon rate absorbed by an atomic conductance G reads simply as $R' = G|V|^2$. Time dependences are denoted by $\exp(-i\omega t)$. Laser-diode intensity fluctuations are conveniently expressed in terms of a relative-intensity noise (N_{ri}).

For convenient reference, let us quote the expression of the angular linewidth $\delta\omega$ of homogeneously broadened lasers [1, 2]. Assuming a linear cold nondispersive absorber

$$\frac{P}{\hbar\omega}\delta\omega = \frac{1+\alpha_0^2}{2(\tau_c + \tau_0)^2}\frac{1+\eta}{2} \qquad \alpha_0 \equiv 2\tau_0(\omega - \omega_0) \qquad (1)$$

where P denotes the optical power transferred from the emitting atoms to the absorber, $\omega - \omega_0$ is the detuning and τ_0^{-1} the gain linewidth. Note that α_0, as we define it, is positive *above* the atomic-transition frequencies, but the sign of α_0 does not matter in equation (1).

The cavity lifetime is

$$\tau_c \equiv \frac{C}{G} = \frac{2L/v_g}{T_1 + T_2}. \qquad (2)$$

The first expression of τ_c in equation (2) is applicable to lumped-circuit resonators of capacitance C and absorbing conductance G. The second expression is applicable to Fabry–Pérot resonators of length L, group velocity v_g and mirror power transmissions T_1 and T_2, respectively, where $T \equiv 1 - R$ if R denotes the power reflection.

The η-parameter is related to the population difference $d \equiv n_e - n_a$, where n_e denotes the fraction of atoms in the emitting state and n_a the fraction of atoms in the absorbing state. We consider two-level atoms with $n_e + n_a = 1$:

$$\eta \equiv \frac{n_e + n_a}{n_e - n_a} \equiv \frac{1}{d} \qquad n_{sp} \equiv \frac{1 + \eta}{2}. \tag{3}$$

If there is a full population inversion η and n_{sp} are unity.

The linewidth formula, equation (17) in section 2, is valid for any collection of emitting and absorbing atoms (or semiconductors) and pumping schemes. In section 3, we consider the response of two-level atoms. For Lorentzian atomic-frequency distributions, the general formula is in equations (35) and (36) of section 4. Special cases are considered in section 5, in particular the large inhomogeneous-broadening limit.

2. Multiple-element laser linewidth

Our laser model is a circuit consisting of any number of admittances $-Y_k(\omega, d_k)$ in parallel, where ω denotes the oscillation angular frequency and d_k the population difference, submitted to voltage V at frequency ω. For each element, a complex emitted photon rate

$$R_k(\omega, d_k, |V|^2) = -Y_k(\omega, d_k)|V|^2 + r_k \tag{4}$$

is defined (the minus sign is introduced so that R' is positive for emitters). The $r_k \equiv r'_k + ir''_k$ are complex white Gaussian-noise rates (not to be confused with the parameter r). From the observation that atoms submitted to prescribed optical fields are independent, it follows that r'_k and r''_k are uncorrelated and have spectral densities equal to $\eta_k R'_k$, where the η_k, describing incomplete population inversion, are given in equation (3) with the label k (primes denote real parts).

Let $P_k(d_k) + p_k$ denote the net pump rate, namely the difference between the pump rate J_k and the spontaneous decay rate S_k. The p_k are net pump-rate fluctuations. The pump 'relative-intensity noise' is denoted by $N_{ri} \equiv (\xi - 1)/J$, where J denotes the mean pump rate and ξJ the spectral density of its fluctuations, with $\xi = 0$ for a quiet pump and $\xi = 1$ for a Poissonian pump. N_{ri} is independent of (cold, linear) attenuations, even if it is negative ($\xi < 1$). If the pump light is split into any number of beams, the cross-spectral densities between p_k and p_ℓ read

$$S^{(p)}_{k\ell} = (J_k + S_k)\delta_{k\ell} + J_k J_\ell N_{ri} \tag{5}$$

where $\delta_{k\ell} = 1$ if $k = \ell$ and 0 otherwise. The pump fluctuations p_k and p_ℓ are partly correlated because elements k and ℓ share the same pump power, except for Poissonian pumps. The above result is readily obtained by considering a transmission line terminated by any number of conductances G_k summing up to the transmission line characteristic conductance. These G_k are endowed with independent noise sources r_k as in equation (4). This formulation [6] is an alternative to the often employed beamsplitter model, with the same end result.

The laser linewidth depends only on low-frequency fluctuations. In that limit, real emitted rates are equal to pump rates for each element k (atom or ion), that is, using equation (4)

$$-G_k(\omega, d_k)|V|^2 + r'_k = P_k(d_k) + p_k. \tag{6}$$

Expanding equation (6) to first order we obtain

$$A'_k + E'_k + R'_k\rho = C_k + p_k \tag{7}$$

where we have set

$$A_k \equiv R_{k\omega}\Delta\omega + r_k \qquad E_k \equiv R_{kd}\Delta d_k \qquad C_k \equiv P_{kd}\Delta d_k \qquad \rho \equiv \Delta|V|^2/|V|^2 \tag{8}$$

where $\Delta\omega$ and Δd_k are small increments of ω and d_k. Subscripts ω and d denote partial derivatives with respect to ω and d_k, respectively. Quantities such as $R_{k\omega}$ refer to mean (or steady-state) values, even though this is not indicated explicitly.

Equation (7) may be written as

$$E'_k = a_k(p_k - R'_k\rho - A'_k) \tag{9}$$

if we introduce the parameters

$$a_k \equiv \frac{1}{1 - C_k/E'_k} = \frac{1}{1 - P_{kd}/R'_{kd}} \tag{10}$$

which play a key role in the linewidth formula. For linear elements (e.g. absorbers) we have $a = 0$, while for semiconductor elements it is appropriate to set $a = 1$ because absorbed pump rates do not depend much on the population difference d. In solid-state lasers, a is close to unity for nearly resonant atoms but vanishes for strongly detuned atoms. This is why such atoms do not contribute importantly to the laser linewidth in spite of their large $1 + \alpha^2$ factors, unless the number N of atoms is large.

Because the laser model considered is an isolated circuit, the total complex emitted rate (including the imaginary rate $R''_c(\omega) \equiv -B_c(\omega)|V|^2$ entering into the resonator) vanishes. To first order, we therefore have

$$0 = \Delta R = \sum(A_k + E_k + R_k\rho) = \sum(A_k + E_k) \tag{11}$$

since the average rate $\sum R_k = 0$.

Separating the real and imaginary parts of equation (11), introducing phase–amplitude coupling factors α_k according to the relation $E''_k \equiv -\alpha_k E'_k$ (or $R''_{kd} \equiv -\alpha_k R'_{kd}$) and using the expression of E'_k in equation (9), we obtain

$$0 = \sum(A'_k + E'_k) = \sum[(1 - a_k)A'_k + a_k p_k] - \rho\sum(a_k R'_k) \tag{12}$$

$$0 = \sum(A''_k + E''_k) = \sum(A''_k + a_k\alpha_k A'_k - a_k\alpha_k p_k) + \rho\sum(a_k\alpha_k R'_k). \tag{13}$$

The unknown relative-intensity fluctuation ρ is eliminated by multiplying equation (12) by the average α-factor

$$\alpha_s \equiv \frac{\sum(a_k\alpha_k R'_k)}{\sum(a_k R'_k)} \tag{14}$$

and adding up equation (13)

$$\sum[A''_k + (\alpha_s + \delta_k)A'_k + \delta_k p_k] = 0 \qquad \delta_k \equiv a_k(\alpha_k - \alpha_s) \tag{15}$$

(δ_k should not be confused with the symbol $\delta_{k\ell}$). Finally, we reintroduce in equation (15) the definition of A_k in equation (8) and obtain $\Delta\omega$ in terms of noise sources of known statistical properties according to

$$\Delta\omega\sum[R''_{k\omega} + (\alpha_s + \delta_k)R'_{k\omega}] + \sum[r''_k + (\alpha_s + \delta_k)r'_k + \delta_k p_k] = 0. \tag{16}$$

Since the linewidth $\delta\omega$ is equal to the spectral density of $\Delta\omega(t)$ for Gaussian processes, it suffices to add the spectral densities of the independent noise sources r'_k, r''_k, given earlier,

with coefficients equal to the squares of their coefficients in equation (16). The general expression for the laser linewidth is therefore

$$\delta\omega = \frac{\sum\{[1 + (\alpha_s + \delta_k)^2]\eta_k R'_k\} + \sum\sum \delta_k \delta_\ell S^{(p)}_{k\ell}}{\left\{\sum[R''_{k\omega} + (\alpha_s + \delta_k)R'_{k\omega}]\right\}^2}. \tag{17}$$

The double sum in the numerator of equation (17) reduces to a sum plus the square of a sum on account of the expression of the cross-spectral densities in equation (5), in which we neglect, for simplicity, the spontaneous (radiative or nonradiative) decay terms S_k. This approximation is a valid one when the pump power exceeds, approximately, ten times the threshold value. To avoid confusion, let us emphasize that neglecting S_k in no way affects the fundamental noise sources $r_k(t)$ responsible for nonzero linewidths. Furthermore, from the expression of α_s in equation (14) and the fact that $J_k = R'_k$ we have $\sum \delta_k J_k = 0$ and thus the term proportional to the pump-laser relative-intensity noise drops out. Therefore, in the present model, the laser linewidth is independent of pump-intensity fluctuations. This conclusion also holds when spontaneous decay is taken into account provided the laser can be tuned at the line centre. In the general case of nonsymmetrical lineshapes and with spontaneous decay accounted for, linewidths may be enhanced by pump fluctuations.

Let us introduce the total real rate $R' \equiv P/\hbar\omega$ transferred from emitting to absorbing atoms, where P denotes the total laser output power and $\hbar\omega$ the photon energy. If the numerator in equation (17) is divided by R' and the denominator by R'^2, we obtain

$$4\frac{P}{\hbar\omega}\delta\omega = \frac{1 + \alpha_s^2 + \{[1 + (\alpha_s + \delta_k)^2]\eta_k + \delta_k^2\}_{\text{av}}}{\tau_p^2}. \tag{18}$$

In the numerator of equation (18) we have extracted the linear loss term labelled '0' (with $a_0 = 0, \delta_0 = 0, \eta_0 = 1$), namely $1 + \alpha_s^2$. In the second term in the numerator, 'av' denotes a sum over *active atoms only* with the R'_k as weighting factors, divided by R'.

The photon lifetime τ_p reads

$$\tau_p \equiv \tau_c + \tau_e^* = \tau_c - \frac{\sum[R''_{k\omega} + (\alpha_s + \delta_k)R'_{k\omega}]}{2R'}. \tag{19}$$

We have extracted from the sum in the denominator of equation (17) the term $R''_{c\omega} \equiv -2R'\tau_c$ relating to the cavity. The cavity lifetime τ_c is given in equation (2) for simple (lumped circuit and Fabry–Pérot) cavity models.

When the α-factors vanish, there is full population inversion ($\eta = 1$), and in the so-called 'good-cavity' limit ($\tau_c \gg \tau_e^*$), the right-hand side of equation (18) is simply $2/\tau_c^2$. This is half the value applicable below threshold, with amplitude fluctuations contributing negligibly to linewidth in that case, as is well known.

Equations (18) and (19) are our final general results. They are applied in the next section, for illustration, to inhomogeneously broadened atomic lasers.

3. Atomic lasers

Let us consider two-level atoms and one-photon processes. Noise terms are ignored in the present section. The admittance of a single atom with transition frequency ω_k submitted to an optical field at frequency ω reads

$$Y_k(\omega, d_k) \equiv G_k + iB_k = -d_k G_0 \frac{1}{1 + i\alpha_k} = -d_k G_0 \frac{1 - i\alpha_k}{1 + \alpha_k^2} \tag{20}$$

where

$$\alpha_k \equiv 2\tau_0(\omega - \omega_k) \qquad d_k \equiv n_{ek} - n_{ak} \tag{21}$$

G_0 denotes the peak conductance of an absorbing atom and τ_0^{-1} is the homogeneous spectral width. The phase–amplitude coupling factor α_k expresses detuning. The absorbing- and emitting-state populations are denoted by n_{ak} and n_{ek}, respectively. The real part G_k of Y_k is negative when the population inversion d_k is positive.

The emitted rate reads, according to equation (20), as

$$R_k \equiv R'_k + iR''_k = -Y_k|V|^2 = d_k I \frac{1 - i\alpha_k}{1 + \alpha_k^2} \tag{22}$$

where $I \equiv G_0|V|^2$ represents a normalized oscillation intensity.

For solid-state lasers and undepleted pumps, it is a good approximation to assume that the pump rate absorbed by atoms is proportional to the absorbing-state population n_{ak}:

$$J_k = \frac{n_{ak}}{\tau_a} = (1 - d_k)I_p \tag{23}$$

where $I_p \equiv 1/2\tau_a$ is a normalized pump intensity.

Note that oscillation or pump 'intensities' are proportional to the modulus squared of the respective fields. Since spontaneous decay is neglected, the absorbed pump rates J_k are equal to the emitted photon rates R'_k. But we are still free to vary the ratio I/I_p by changing the coupling of the pump light to the active atoms.

Solving for d_k the relation $J_k = R'_k$ with the help of equations (22) and (23) we obtain

$$d_k = \frac{1}{\eta_k} = \frac{1 + \alpha_k^2}{1 + I/I_p + a_k^2} \equiv \frac{q^2 + x_k^2}{1 + x_k^2} \tag{24}$$

where we have defined $q \equiv 1/\sqrt{1 + I/I_p}, x_k \equiv q\alpha_k$. Using equation (24), equation (22) reads

$$R_k \equiv R'_k + iR''_k = I\frac{q^2 - iqx_k}{1 + x_k^2}. \tag{25}$$

The a-factor is defined in equation (10), where C_k and E'_k follow from the definitions in equation (8). The expression of $R_{k\omega}$ is obtained from equation (22) and that of $P_{kd} \equiv J_{kd}$ from equation (23), by taking the derivative with respect to d_k. We obtain

$$a_k \equiv \frac{1}{1 - J_{kd}/R'_{kd}} = \frac{1 - q^2}{1 + x_k^2} = 1 - d_k. \tag{26}$$

Equations (24)–(26) give explicit expressions for the parameters R'_k, d_k, η_k and a_k that enter in the general expression in equations (18) and (19), in terms of x_k.

In the next section we consider a particular distribution of atomic transition frequencies ω_k, namely a Lorentzian one, selected for the sake of mathematical simplicity. For some lasers, and particularly gas lasers, a Gaussian distribution would perhaps be more appropriate. But no qualitative differences are expected for these two distributions.

4. Lorentzian transition-frequency distribution

The sums over k in equations (18) and (19) can be converted into integrals considering the very large number of atoms involved. Let the number dN of atoms whose normalized transition frequencies are between α_i and $\alpha_i + d\alpha_i$ be

$$dN = \frac{N}{\pi}\frac{d\alpha_i}{1 + \alpha_i^2} \qquad \alpha_i < 2\tau_i(\omega_k - \omega_i) < \alpha_i + d\alpha_i \tag{27}$$

where the subscript 'i' stands for 'inhomogeneous'. The parameter τ_i^{-1} is the inhomogeneous distribution linewidth and ω_i the centre frequency. Integration (with the lower limit extended to minus infinity because relatively narrow spectral widths are considered) shows that N is the number of atoms.

Let us introduce the ratio r of the inhomogeneous-to-homogeneous spectral-width ratio and the normalized parameter g,

$$r = 1 + \frac{\tau_0}{\tau_i} = \frac{\tau_0}{\tau_e} \qquad g - 1 \equiv q(r - 1) \tag{28}$$

where τ_e^{-1} represents the inhomogeneous spectral width, and a detuning parameter δ (not to be confused with δ_k) normalized by the inhomogeneous spectral width and normalized parameter y,

$$\delta \equiv 2\tau_e(\omega - \omega_i) \qquad y \equiv qr\delta = q2\tau_0(\omega - \omega_i). \tag{29}$$

With the help of the normalized transition frequency distribution

$$w(x) \equiv \frac{(g - 1)/\pi}{(g - 1)^2 + x^2} \tag{30}$$

summations can be written as integrals according to the rule

$$\sum f(x_k) \to N \int_{-\infty}^{\infty} dx\, w(x - y) f(x). \tag{31}$$

With these definitions the total real emitted photon rate is obtained from equation (25) by contour integration

$$R' \equiv \sum R'_k = NIq^2 \int_{-\infty}^{\infty} dx \frac{w(x - y)}{1 + x^2} = \frac{NIq^2}{\gamma} \qquad \gamma \equiv \frac{g^2 + y^2}{g}. \tag{32}$$

The oscillation condition requires that the emitted rate be equal to the absorbed rate. Let N_0 be the number of atoms permitting oscillation at infinite pump intensity (corresponding to full population inversion). Equating the expression in equation (32) with the same expression with N changed to N_0 and $q = 1$, $g = r$, we obtain

$$n \equiv N/N_0 = \frac{g^2 + q^2 r^2 \delta^2}{q^2 gr(1 + \delta^2)} \qquad g = 1 + q(r - 1). \tag{33}$$

This expression shows that q can be expressed as a function of our three basic parameters: n, r and δ as the solution of a third-degree equation.

The expression of α_s in equation (14) is converted into the ratio of two integrals that can be evaluated by contour integration

$$s \equiv q\alpha_s = \frac{2gy}{g^2(g + 1) + y^2(g - 1)}. \tag{34}$$

In terms of the normalized parameters and after much rearranging (in particular, the term '$1 - a$' introduced in the denominator of the integrand by the η-factor is found to cancel out, so that the only poles are those of $a(x)$ and $w(x - y)$) the linewidth is given by

$$4(\tau_c + \tau_e^*)^2 \frac{P}{\hbar\omega}\delta\omega = 1 + \frac{1 + 2s^2}{q^2} + \frac{q^2 - 1}{q^2}\gamma \int_{-\infty}^{\infty} dx\, w(x - y)\frac{(x - s)^2(x^2 + q^2)}{(1 + x^2)^3} \tag{35}$$

where $\gamma \equiv (g^2 + y^2)/g$.

The effective atomic lifetime τ_e^* is obtained by derivation of R'_k and R''_k in equation (25) with respect to ω. Rearranging gives

$$\frac{\tau_e^*}{\tau_0} = 1 - \gamma \int_{-\infty}^{\infty} dx\, w(x - y)\frac{2x(x - s)(x^2 + q^2)}{(1 + x^2)^3}.$$

Equations (35) and (36) give the power–linewidth product in terms of the three basic parameters n, r and δ. These equations can be written in closed form by contour integration, but the resulting expressions would not be very helpful. We therefore leave them as integrals to be evaluated numerically.

5. Special forms and numerical results

In the homogeneous limit $r = 1$, $g = 1$, the $w(x)$ function reduces to the $\delta(x)$ distribution and equation (35) simplifies to

$$(\tau_c + \tau_0)^2 \frac{P}{\hbar\omega}\delta\omega = \frac{1 + \delta^2}{2}\frac{1 + n}{2}. \tag{37}$$

This is the Haken–Lax result in equation (1) because in the present notation $\alpha_0 = \delta$ and $\eta = n$.

Next consider inhomogeneously broadened lasers with $n = 1$. Then $q = 1$, $g = r$, $a = 0$ and $y = r\delta$. The last term of equation (35) vanishes and therefore in the good-cavity limit

$$\tau_c^2 \frac{P}{\hbar\omega}\delta\omega = \frac{1 + s^2}{2} \tag{38}$$

where

$$s = \frac{2\delta}{r + 1 + \delta^2(r - 1)} \approx \frac{2\delta}{r} \qquad r \gg 1 \qquad \delta \ll 1. \tag{39}$$

This expression is essentially the one reported by Haken [1]. The approximations made in that reference therefore appear to be essentially equivalent to assuming that $N \approx N_0$ in our notation. That is, it is assumed that the number of active atoms does not much exceed the minimum required value.

In the strongly inhomogeneous limit, $r \to \infty$, but without detuning ($\delta = 0$), we have $q = 1/n$, and we obtain

$$\tau_c^2 \frac{P}{\hbar\omega}\delta\omega = \frac{1/n^2 + 10 + 5n^2}{32}. \tag{40}$$

In the large-n limit, the inhomogeneous-to-homogeneous laser-linewidth ratio is thus $5n/8$.

In the same limit, but with finite r-values, this ratio is [7]

$$\frac{\text{inhomogeneous laser linewidth}}{\text{homogeneous laser linewidth}} = \frac{\text{inhomogeneous spectral width}}{\text{homogeneous spectral width}} \equiv \frac{\tau_0}{\tau_e}.$$

The general solution is illustrated in figure 1 for $r = 10$ and $n = 10$ in the good-cavity limit. In that example, the ratio τ_e^*/τ_0 is equal to 0.68 when $\delta = 0$ and 0.525 when $\delta = 3$.

When $\delta = 0$, r is large, but spontaneous decay is taken into account, with a time constant τ_s (i.e. $S_k = n_{ek}/\tau_s$) the expression is (the detailed calculations will be omitted)

$$\tau_c^2 \frac{P}{\hbar\omega}\delta\omega = \frac{c/n^2 + 8 + 2/c + (8c^2 - 3)n^2/c^3}{32} \tag{41}$$

where $c \equiv (\tau_s/\tau_a + 1)/(\tau_s/\tau_a - 1)$ is supposed to be the same for all the atoms. As before, τ_a represents the pump rate time constant, according to equation (23). This expression in equation (41) reduces to equation (40) when spontaneous decay is neglected ($c = 1$). The difference between equations (40) and (41) does not exceed 10% when $\tau_s/\tau_a > 5$.

6. Conclusion

We have presented a semiclassical theory of inhomogeneously broadened laser linewidth which is simple, at least in principle. A closed-form expression for the linewidth enhancement caused by inhomogeneous broadening has been reported in equation (17), which is general, except for the assumption that the atoms are submitted to the same relative field-intensity fluctuations. This assumption is valid for high-reflectivity mirrors.

According to this theory, the linewidth of cold solid-state lasers, for example, should be significantly increased by inhomogeneous broadening when the number of ions N much exceeds the threshold value N_0. If, for example, the laser is constructed with $N = 10N_0$ inhomogeneous broadening enhances the power–linewidth product by a factor of 6. For large N values, linewidth enhancement is limited to the inhomogeneous-to-homogeneous spectral-width ratio. Further, we have found that for the particular pump model considered (pump rates proportional to the lower-state populations) the linewidth does not depend on pump fluctuations, even in the case of detuning between the cavity resonant frequency and the centre of the atomic-transition frequency distribution. The differences between the conclusions reached here and in the recent paper by Khoury *et al* [4] are attributed to different assumptions concerning the manner by which atoms are pumped.

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