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Classroom Demonstration of the Laws of Propagation of Gaussian Beams

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A method for simulating the propagation of coherent optical beams with Gaussian irradiance patterns in free space and through unaberrated lenses is described. This method is based on the skew-ray representation of Gaussian beams. The rotation in space of a collimated laser beam with skew axis generates a Gaussian beam profile. The phase of the optical field is given by the angular position of the laser-beam center.

During the last decade, coherent optical beams with Gaussian irradiance patterns have been found to be of particular interest because the irradiance patterns of such beams are preserved, except for a scaling factor, as the beams propagate in free space or through unaberrated lenses.[1] Furthermore, most modern lasers generate optical beams whose irradiance is almost (though not quite) Gaussian.

The most obvious way of demonstrating the laws of propagation of Gaussian beams is to make the beam generated by a single mode He–Ne laser visible with the help of a moving screen. The beam parameters can be measured accurately with a chopper and a detector.[2] This direct approach, however, is not very satisfactory for classroom demonstration for a number of reasons. First of all, at visible wavelengths ($\lambda \sim 0.6\ \mu$m), a beam whose minimum cross-section can be seen at a distance (e.g. a beam with a minimum radius of 6 mm) has a very small angular divergence, of the order of 10^{-4} rad. Therefore, no significant change in beam radius occurs in free space over the length of a classroom. Inversely, if the divergence of the beam is made large by focusing the beam to a small spot size with the help of a microscope objective, the beam waist itself is too small to be observed directly. A second difficulty is that the irradiance patterns of beams generated by most lasers are not quite Gaussian. The sharp edge apertures incorporated in the cavity to prevent oscillation on higher order transverse modes significantly perturb the irradiance pattern of the fundamental mode. In the case of confocal mirrors, for instance, the fundamental mode of resonance is described by a generalized prolate spheroidal wave function, which exhibits oscillations in the tail of the irradiance pattern.[1] Some improvement could be obtained on that respect by using spatial filtering techniques. These techniques, however, are not easy to implement. Finally, it should be noted that the phase of the optical field can be measured only through delicate interferometric or heterodyning techniques.

We shall describe in this paper an easily implemented method of simulating Gaussian beam propagation. This method requires the use of a He–Ne laser and a pair of rotating lenses. The characteristics of the laser beam are unimportant as long as the beam has small angular divergence and does not depart too much from a geometrical optics ray.

Let us first recall a few essential facts concerning the propagation of two-dimensional monochromatic Gaussian beams through isotropic lossless square-law media. The refractive index of such media varies with the transverse (x) and axial (z') coordinates according to a law of the form

$$n(x, z') = n_0(z') - \tfrac{1}{2} n_0(z')^{-1} \Omega^2(z') x^2, \qquad (1)$$

where $n_0(z')$, $\Omega^2(z')$ are arbitrary real functions of z'. It is convenient to introduce a reduced axial coordinate z, defined by

$$z \equiv \int_0^{z'} n_0(z')^{-1} dz',$$

and a reduced field

$$\psi(x,z) \equiv n_0(z')^{1/2} E(x,z') \times \exp\left(-ik \int_0^{z'} n_0(z') dz'\right) \quad (2)$$

where $k \equiv \omega/c$ denotes the free-space propagation constant and $E(x,z')$ the magnitude of the electric field, assumed linearly polarized. An $\exp(-i\omega t)$ time dependence of E is understood. Note that the beam irradiance is proportional to $n_0 EE^* = \psi\psi^*$. In the special case where the on-axis refractive index n_0 is unity, we have $z' = z$ and $E = \psi \exp(ikz)$.

Within the approximation of Gauss, the reduced field of a Gaussian beam at a point x, z, is given by the simple expression[3]

$$\psi(x,z) = q(z)^{-1/2} \exp[i(k/2)\dot{q}(z)q(z)^{-1}x^2], \quad (3)$$

where the upper dot denotes differentiation with respect to z, and $q(z)$ is a solution of the paraxial ray equation

$$\ddot{q}(z) + \Omega^2(z) q(z) = 0. \quad (4)$$

A lens with focal length f is equivalent to a thin section of square-law medium with focusing constant Ω^2 and thickness Δz such that $\Omega^2 \Delta z = f^{-1}$, as one finds by equating the transverse variations in optical thickness of the lens and square-law medium. Thus, Eq. (3) is applicable to arbitrary sequences of converging ($\Omega^2 > 0$) or diverging ($\Omega^2 < 0$) lenses, as well as to lens-like media. In free space we have, of course, $\Omega^2 = 0$.

When $q(z)$ is a real solution of Eq. (4), $\psi(x,z)$ represents the field of a ray pencil. The physical significance of Eq. (3) is then clear: the exponential term expresses, within the paraxial approximation, the departure of the (spherical) wavefront from the tangent plane. The prefactor $q(z)^{-1/2}$, on the other hand, is a consequence of the law of conservation of power. Equation (3) describes Gaussian beams when complex initial values are given to $q(z)$. Ω^2 being real for lossless media, both the real and imaginary parts $q_r(z)$ and $q_i(z)$ of $q(z)$ obey Eq. (4) and behave like ordinary rays. The complex ray $q(z)$ can therefore be represented by a real skew ray (i.e., a ray that does not intersect the z axis) in the three-dimensional space q_r, q_i, z.

On axis, the reduced field is [setting $x=0$ in Eq. (3)],

$$\psi(0,z) = q(z)^{-1/2}. \quad (5)$$

Thus, the phase $\theta(z)$ of the on-axis reduced field is equal to minus one-half the phase of $q(z)$:

$$\theta(z) = -\tfrac{1}{2} \tan^{-1}[q_i(z)/q_r(z)]. \quad (6)$$

This phase term is important mainly because it defines the resonance frequencies of open resonators. In the skew-ray representation of Gaussian beams, the phase of $q(z)$ is the angle that the skew ray makes with the q_r axis at plane z. The phase of the optical field can therefore be given a simple geometric representation.

It is not difficult to show that if the complex ray $q(z)$ is normalized by the (z-invariant) condition

$$q_r(z)\dot{q}_i(z) - q_i(z)\dot{q}_r(z) = k^{-1}, \quad (7)$$

the beam halfwidth $\xi(z)$, defined as the distance from axis where the beam irradiance $\psi\psi^*$ has dropped by a factor $e = 2.718\ldots$, is equal to the modulus of $q(z)$; we have

$$\xi(z) = [q(z)q^*(z)]^{1/2} = \{[q_r(z)]^2 + [q_i(z)]^2\}^{1/2}. \quad (8)$$

The beam halfwidth is equal, at any plane z, to the distance between the skew ray and the z axis. By letting the skew ray rotate about the z axis, the beam profile is therefore generated. In particular, in free space the skew ray is a straight line whose rotation generates an hyperboloid of revolution.

For Gaussian beams with rotational symmetry, the reduced field is, from Eq. (3),

$$\Psi(r,z) = \psi(x,z)\psi(y,z) = q(z)^{-1} \exp[i(k/2)\dot{q}(z)q(z)^{-1}r^2], \quad (9)$$

where $r^2 \equiv x^2 + y^2$. The beam irradiance pattern in a meridional plane is therefore the same as before. The phase of the on-axis reduced field, however, varies twice as fast as a function of z.

Note that if the lens system under test were misaligned, the simulated beam would remain an exact scaling of the real beam. In both cases the beam center follows a real ray trajectory.[3]

The expression given in Eq. (3) for the field of a Gaussian beam is easily generalized to the case of nonorthogonal astigmatic optical systems and to the case of anisotropic media, provided the ray optics Hamiltonian remains at most quadratic in its arguments (approximation of Gauss). To obtain these generalizations we essentially have to replace q by a 2×2 matrix Q which formally obeys the ray equation, and $\dot{q}$ by the quantity canonically conjugate to q (see Ref. 4). The skew ray representation, however, becomes complicated in those cases. We shall therefore restrict ourselves to two-dimensional Gaussian beams and to Gaussian beams with rotational symmetry.

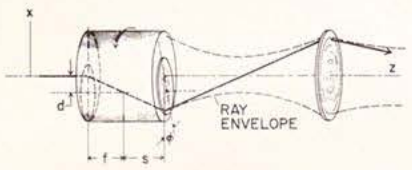

Fig. 1. Schematic diagram of demonstration system showing, on the left, a rotating device incorporating two off-set lenses. The skew laser beam axis simulates Gaussian beam propagation through a test lens (shown on the right).

Let us now indicate how a rotating skew ray can be generated in practice and how the experiment can be implemented. Consider two lenses with focal length f separated by a distance $2f$. Let their common axis be off-set by a distance d from the z axis, and the system (shown on the left of Fig. 1) be made to rotate about the z axis at some convenient speed, perhaps 120 rpm. If a narrow laser beam (which represents a ray in this demonstration) enters in the lens system along the rotation axis, the outgoing ray remains parallel to its original direction while rotating with a radius $2d$. Now let the separation between the lenses be $f+s$, $s>f$, instead of $2f$. As the lens system rotates, the outgoing ray, which now crosses the z axis, generates a cone which can be viewed as a converging ray pencil. In order to simulate diffraction effects some skewness need be introduced in the outgoing ray. This can be done by rotating the axis of the second lens with respect to the first lens by an angle ϕ, as shown in Fig. 1.

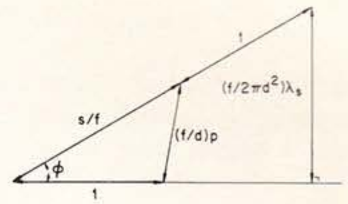

Fig. 2. The wavelength λ_s and angular divergence p of the simulated Gaussian beam are obtained from the geometrical construction shown on this figure. The minimum beam halfwidth is related to p and λ_s by $\xi_0 = \lambda_s/2\pi p$; s, f, d, and ϕ are the lens system parameters shown in Fig. 1.

The minimum halfwidth ξ_0 of the simulated beam, its far-field angular divergence

$$p \equiv \lim_{z\to\infty} (\xi/z),$$

and its wavelength $\lambda_s = 2\pi p \xi_0$, are easily obtained from the laws of Gaussian optics. These quantities are found to be related to the parameters s and ϕ defining the relative position of the two lenses by the geometrical construction shown in Fig. 2.

We find, using this construction, that if $f=10$ cm, $s=13$ cm, $d=1$ cm, and $\phi=30°$, the diffraction effects are the same as if the wavelength were equal to 7.3 mm (more than ten thousand times the actual wavelength of the laser) corresponding to a frequency of 40 GHz. The beam waist radius

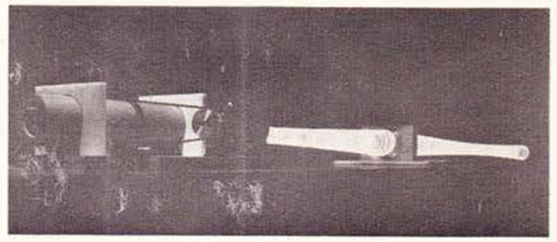

Fig. 3. Photograph of the experiment. The simulated beam is made to converge with the help of a lens. It subsequently diverges as a result of the (simulated) diffraction effects.

is $\xi_0 = 18$ mm and the angular divergence is $p = 3.8°$.

Figure 3 shows a photograph of the experiment. The rotating skew ray has been made visible by slowly moving a screen along the system axis during the time of exposure of the photograph. For the set of parameters chosen in this experiment, the simulated Gaussian beam diverges at the exit of the rotating device. The beam is subsequently refocused with the help of a test lens, reaches its waist about 1 m away from the test lens, and diverges again. Almost any beam waist radius and wavelength can be simulated by changing the axial and azimuthal position of the second lens with respect to the first.

The device that we have described can also be used in courses of quantum mechanics to demonstrate the spreading in time of minimum uncertainty wave packets. The analogy existing between nonrelativistic quantum mechanics and scalar optics is well known (see, for instance, Ref. 5) and need not be insisted upon here.

ACKNOWLEDGMENTS

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[1] H. Kogelnik and T. Li, Appl. Opt. **5**, 1550 (1966).

[2] J. A. Arnaud, W. M. Hubbard, G. D. Mandeville, B. la Clavière, E. A. Franke, and J. M. Franke, Appl. Opt. **10**, 2775 (1971).

[3] J. A. Arnaud, Appl. Opt. **8**, 1909 (1969); for a detailed historical account, see Ref. 4.

[4] J. A. Arnaud, "Hamiltonian Theory of Beam Wave Propagation," in *Progress in Optics*, Vol. XI, edited by E. Wolf (North-Holland, Amsterdam, to appear).

[5] R. P. Feynmann, Rev. Modern Phys. **20**, 367 (1948).