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TUTORIAL REVIEW

Classical theory of laser noise

J. ARNAUD

Université de Montpellier II, Equipe de Microoptoélectronique de Montpellier, Unité associée au CNRS 392, USTL, Pl. E. Bataillon, 34095 Montpellier Cédex 2, France

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The classical theory of laser noise treats light in a classical manner, yet agrees with quantum theory for large particle numbers. The basic concept is that laser noise is caused by atomic jumps between lower and upper levels, and that atoms subjected to classically-prescribed optical fields are independent. The treatment of amplitude noise of single-mode cavities containing resonant three-level atoms is applicable to semiconductor lasers at moderate power. At high power one must account for the dependence of the gain on optical power and for state-occupancy fluctuations. The phasor theory that attributes noise to the beat between the oscillating field and the field spontaneously emitted in the mode by excited-state atoms cannot be understood consistently in semiclassical terms.

1. Introduction

The classical theory of laser-light amplitude and phase noise was proposed by this author in 1988[1] [1–6]. According to this theory, fluctuations result from atomic jumps between lower and upper levels, and atoms subjected to prescribed optical fields are independent. Light is described in a classical manner, i.e. by means of commuting functions of time. Accordingly, the words 'photon number' and 'photon rate' are employed in this paper only as short-hand for electromagnetic energy and electromagnetic power, respectively, divided by $h\nu_0$, where $h \approx 6.6 \times 10^{-34}$ J s denotes Planck's constant, and ν_0 the mean oscillation frequency. The classical theory of noise agrees exactly with quantum theory for large particle numbers, even when the so-called 'nonclassical' states of light are generated.

Only stationary amplitude noise is treated to meet space limitations. It is permissible to ignore phase fluctuations when both the emitting and absorbing elements are frequency-insensitive, e.g. for index-guided laser diodes operating near peak-gain in the absence of spurious optical reflections. Expressions are derived for familiar quantities such as the relative intensity noise (RIN), with special attention given to nonfluctuating (or 'quiet') pumps. The

[1] This early paper, though correct for the situation considered, contains misleading statements, corrected in [2]. Firstly, a doubled noise source intensity is attributed to the emitter and no noise to the absorber. This unsymmetrical solution is accurate only in the linear regime. Secondly, the implication that the field factorizes into the product of a function of space and a function of time is in general erroneous. The independent noise sources distort the field distribution.

results apply to resonant three-level atoms, but a cursory discussion of laser diodes is also offered. It is recalled in the appendix why spectral densities cannot be negative.

Let us first explain what is meant by 'optical detection'. When a detector such as a p-i-n junction is subjected to light, e.g. laserlight or starlight, the detector current consists of a series of pulses whose areas are equal to the electron charge, occurring at times t_k, $k = 1, 2, \ldots$. If the optical power is, say, 1 mW and the free-space wavelength is 1.55 μm, the average photon rate is $8 \times 10^{15}\,s^{-1}$. Such high rates can be recorded only with a mosaic of small-area detectors, because optical attenuators aimed at reducing the average rate to practical values would 'kill' photons randomly and spoil the light-source statistics. The full information is given by the probability $p(t_1, t_2, \ldots, t_n)$ that photoelectron events occur at those times. When the photoelectron occurrence times are independent of one another, the distribution is called Poissonian, and the probability factorizes as $p(t_1)\, p(t_2) \cdots p(t_n)$.

In most practical cases the detector response time, of the order of 10 ps, and thermal noise prevent us from identifying individual photoelectrons. The detector photocurrent then has the appearance of a current $\langle i \rangle$ of the order of 1 mA, plus some small zero-mean fluctuation $\Delta i(t)$. A crude but often sufficient way of characterizing $\Delta i(t)$ is through its covariance, the time-averaged product of $\Delta i(t)$ and $\Delta i(t+\tau)$. Alternatively, a radiofrequency analyser provides the photocurrent spectrum, which is the Fourier transform of the covariance. The detector-load thermal noise as well as subsequent electronic amplifier noise can be subtracted out, and arbitrarily small noise levels can be measured, in principle, by increasing the integration time. The current from an ideal detector exactly reflects the incident light fluctuations. No 'detector shot-noise' should be considered. It will be assumed, as usual, that nonideal detectors are equivalent to ideal detectors preceded by some optical loss.

In place of the detected electrical current, it is convenient to consider the photoelectron rate $Q \equiv i/e \equiv \langle Q \rangle + \Delta Q$. The 'instantaneous' rate Q may be defined as the reciprocal of the appropriately smoothed delay between successive photoelectrons. For a Poissonian distribution the (double-sided) spectral density of ΔQ is equal to the average rate $\langle Q \rangle$. But when the photoelectron occurrence times are evenly spaced, ΔQ vanishes.

Laser sources are characterized by the optical cavity employed, and the way the enclosed atoms are raised to the upper state by a process called 'pumping', which may be random or regular. Key theories describing above-threshold amplitude fluctuations are now considered. Background information on lasers may be found in Siegman's comprehensive book [7]. An account of maser and laser history, and discussions of the concept of photon and semiclassical theories can be found in [8].

1938: Classical oscillator theory ($kT \gg h\nu$)

Johnson discovered experimentally in 1928 that electrical resistances generate noise. This noise, observed with the help of large-gain amplifiers, can be represented by a white-voltage source of spectral density $2kTR$, where k denotes Boltzmann's constant and T the absolute temperature of the resistance R. This experimental result was soon explained by Nyquist [9]. Generalization of the Johnson–Nyquist result to the quantum regime, with or without electromagnetic zero-point energy, is not free of difficulties, as recently discussed by Gardiner [10].

Classical oscillators may employ a tuned circuit connected between the cathode and the grid of a triode, oscillation being sustained with the help of some feedback from the plate current. Nyquist's noise is the fundamental source of noise of classical oscillators. Building on previous work by Van der Pol, Bernstein [11] set up in 1938 a Langevin equation for oscillators of that kind, and solved the associated Fokker–Planck equation.

In some circumstances fluctuations due to the corpuscular nature of electricity add noise. This noise in fact dominates Nyquist's noise when the tuned circuit is connected to the triode plate, and cathode emission is limited by temperature rather than by space-charge effects. To avoid confusion, let us point out that this shot-noise fluctuation acts directly at the oscillation frequency. Injected-current fluctuations discussed later in connection with lasers correspond instead to rather slow changes in time of a parameter, i.e. the pump power. Oscillator noise differs vastly in nature from the output of narrowband linear amplifiers. For a recent treatment of practical transistor oscillators, see, for example, Braun and Lindenmeier [11].

From a fundamental (i.e. quantum) point of view, classical oscillators should be viewed as multiphotonic devices. Indeed, in a typical classical oscillator such as a reflex klystron, the energy lost by each electron, in the eV range, is considerably larger than the microwave photon energy, which is in the meV range.

1967: Quantum theory of laser noise with random pumping

Early measurements of laser noise have been reported by Freed and Haus, and Armstrong and Smith [12]. Glauber [13] was the first to point out that laser oscillators are very much like classical oscillators in the sense that amplitude fluctuations are small compared with the mean amplitude: photon-number variance is of the order of the mean while, for frequency-filtered thermal light, the variance is the squared mean.

Quantum electrodynamics (QED) is in principle capable of predicting the probability that some outcome will be observed at time t if the source preparation at $t = 0$ is specified. But the full quantum theory is so difficult that exact solutions are known, essentially, only for a single atom and a single-mode cavity. When many atoms are present simultaneously, coarse-graining approximations must be introduced. Quantum theories of laser noise were given in 1967 by a number of theorists: McCumber, Haken, Lamb, Scully, Lax, and others. They obtained the statistics of the photon number for plausible laser-oscillator models, including saturation. This theoretical work, and application to semiconductor lasers, was reviewed in an excellent paper by Haug [14]. The nature of laser noise is described by Morgan and Adams [15] in these terms: 'A fundamental noise source in the output of continuous-wave lasers is the quantum shot-noise due to the random electron transitions which generate laser emission.' We adhere to this viewpoint, stimulated absorption and emission both being considered as sources of noise.

1984: Quiet-pump theory

The earlier theories implicitly assumed Poissonian pumping because this was the practical situation at the time. It was implicit in these works that detected photoelectron fluctuations reflect those of the intracavity field. This, however, is in general not the case.

Golubev and Sokolov [16] were the first to make the observation (rather obvious in retrospect) that if the pump does not fluctuate, and the laser-detector system is ideal, the detected current should not fluctuate at low frequencies. A similar theory was proposed independently in 1986 by Yamamoto and others [17], who furthermore pointed out that when a laser diode is driven by a battery in series with a (preferably cold) resistance much larger than the diode dynamic resistance, the pump (i.e. the driving current) is nonfluctuating. This conclusion follows from Nyquist's theorem, which applies not only to thermodynamic equilibrium, but also when a steady electrical current flows, as long as Ohm's law is applicable [18]. To avoid confusion, let us recall that currents flowing through p-n junctions are known to fluctuate at the shot-noise level when the diode is *voltage* driven. Obviously the current fluctuations of current-driven diodes are determined by the driver, not by the diode. More recent reviews of quantum theories considering reduced pump fluctuations are found in [19]. Experimental results demonstrating sub-Poissonian outgoing-photon statistics (by up to a factor of 10) are listed in [20].

1987: Classical theory of laser noise

Let us first explain in classical terms why nonfluctuating pumps entail nonfluctuating photon flows under ideal conditions at low frequencies, with a picturesque model: Consider a small-sized room with an entrance door and an exit door, and assume that one person enters every second. An hour later, obviously, 3600 persons have entered. Almost exactly 3600 persons have also left the room, because not many people can stay in the room. The exit rate, though not strictly constant, is 'sub-Poissonian'. This is even more so if longer time periods are considered, e.g. one day. This picture applies to electrons injected into a laser diode and extracted from the detector. In the real physical situation one must consider storage of both the electrons and the photons. The closed-room model implies of course that no electrons are lost through spontaneous-recombination processes and that no photons are lost as a result of optical losses, either internally or during propagation from the laser to the detector.

It is interesting to consider also the behaviour of the number of photons m in the optical cavity, following the discussions in [16] and [4]. Since the photon-absorption process is intrinsically random, it may happen that the absorbed photon rate exceeds (say) the average value over some period of time. This causes m to decrease after a while, assuming for simplicity that the photon-generation rate does not fluctuate. Accordingly, the deterministic term in the outgoing photon rate, which is proportional to m, is reduced and the excess rate initially assumed becomes smoothed out. Complete quieting is possible.

Theories such as the one given in this paper that do not quantize the optical field are usually labelled 'semiclassical'. However, earlier semiclassical theories do not enforce energy conservation and are unable to describe 'nonclassical' states of light (see Mandel [8]).

A classical theory of amplitude noise was given by Katanaev and Troshin [21] in 1987, on the basis of Golubev and Sokolov's quantum results. Their theory is accurate. However, it gives no hint concerning a possible treatment of phase noise. Yet, in general, phase and amplitude fluctuations cannot be treated separately. The handling of noise sources in [21] is complicated, particularly in relation to linear optical losses.

Initially [1] this author was mostly concerned with phase noise, and unaware of related quantum-theory results. By enforcing energy conservation, the earlier findings in [16] concerning amplitude-noise squeezing were, however, recovered in a classical manner. In [1–3] the intrinsic noise is attributed to the zero-point fluctuation of the optical field, which is a consequence of optical-field quantization. Our present view is that field quantization is unnecessary for the kind of problem considered. It suffices to observe that atoms (with nonoverlapping wavefunctions) submitted to prescribed classical fields are independent. These concepts lead to the basic formula in Equation 3. This equation tells us that the absorbed photon rate Q is the sum of a deterministic term m/τ_p, proportional to the photon number m, and a fundamental fluctuation q, which is at the shot-noise level for small amplitude fluctuations, i.e. the spectral density of q is equal to the average rate $\langle Q \rangle$. A similar result applies to emitters, except that the deterministic term may then depend also on the carrier number. Emitters and absorbers are treated on the same footing, as done earlier in Lax's semiclassical paper [22].

The classical theory of noise that we present appears to be the limit of linearized symmetrically-ordered quantum theories. Quantum theories employing Glauber's P-distribution may not

be more accurate since the probability distributions obtained are negative, and therefore unphysical, for m-values that depart significantly from the mean value $\langle m \rangle$ [19].

We restrict ourselves to above-threshold lasers. Below-threshold (linear) operation is straightforward. There is a narrow range near threshold where difficulties arise but they do not seem to be of a fundamental nature.

Let us now consider other proposals for explaining laser noise in classical terms. Important textbooks [23] have popularized a 'phasor' picture of phase and amplitude noise. Laser noise would be caused by the addition to the strong classical field of a randomly-phased field, attributed to spontaneous emission, the 'photon events' occurring at an average rate equal to the population inversion factor divided by the lifetime of the photon in the cold cavity. The laser dynamics reacts against amplitude fluctuations while the phase diffuses freely. This phasor picture has been formalized by Henry in a series of papers [24] that provide many insights, in particular in relation to laser-diode linewidth enhancement. But in our opinion the phasor picture is inconsistent, see Section 7.

A classical theory of parametric-oscillator noise by Reynaud and others (for a recent formulation see [25]) exhibits exact agreement with quantum results. Section 3 of [25], dealing with conventional lasers, is, however, inaccurate. As Nilsson's and others recognize (see chapter 3 of the book in [24]), quiet pumps cannot be modelled by suppressing the dipole noise. Amplitude fluctuations may be correctly predicted in that way, but not phase fluctuations. Other assumptions made concerning dipole noise are 'ad hoc'.

Many popular texts assert that shot-noise fluctuations should be associated with any particle flow. This is a misinterpretation of McCumber's quantum results (see the review in [14]), who has shown that shot-noise fluctuations should be associated with atom-to-photon or photon-to-atom *conversions*. For a consistent application of McCumber's results one must consider not only the photon generation mechanism, but also the absorption of photons in the detector. The McCumber shot-noise term for this absorption process is precisely our noise term q, which enters into both the rate equation for the photon number m and the rate equation for the detector carrier number, say n_0. In other words, the theory of laser amplitude noise advocated in this paper may be viewed as a consistent application of McCumber's principles.

Organization of the paper

It is shown in Section 3 that the RIN is not affected by optical attenuators, even if negative, a situation that may occur with quiet pumps. Furthermore, the cross-spectral density between photon-number fluctuation and relative photon noise is found to be independent of attenuation.

High-power laser-oscillators, treated in Section 4, are exceedingly simple because the atomic number need not be considered. One proves that nonfluctuating pumps entail nonfluctuating light at small baseband frequencies, in agreement with previously cited quantum theories. A more realistic oscillator model that takes into account spontaneous recombination is treated in Section 5. The expression for above-threshold low-frequency RIN is derived. The cursory discussion in Section 6 shows that laser diodes are not very different from three-level-atom lasers as long as the intraband scattering time is small, say less than 0.2 ps, and at moderate power. For larger values one must take into account both the spectral-hole-burning effect and statistical fluctuations (derived from equilibrium thermodynamics). Because this paper is tutorial in nature, we avoid discussing intricate, perhaps more realistic, models.

Sans-serif letters are employed to denote gain G, double-sided spectral density S, (non-measurable) light intensity I, and noise sources q, r, s. The averaging signs $\langle\ \rangle$ are omitted when no confusion with instantaneous values may occur, e.g. in the expression of spectral densities. Complex quantities refer to nonzero baseband frequencies f.

2. Absorbers and emitters

The theory treats absorbers and emitters on the same footing. Cold absorbers consist of a large number of two-level atoms, essentially all of them in the ground state. Light at frequency ν_0 is absorbed if $h\nu_0$ is approximately equal to the energy difference between two atomic levels (resonant one-photon transition). Likewise, emitters with complete population inversion consist of atoms in the upper state. In either case the probability that a transition occurs is proportional to the number $m \gg 1$ of photons in the cavity.

Consider first absorbers, e.g. detectors. The transition of an atom from the lower (or 'absorbing') state to the upper ('emitting') state can be monitored in various ways. In optical detectors, atoms in the upper state become ionized and the freed electrons generate an electrical current that can be amplified with the help, for example, of a transistor. When the atomic wavefunctions are neither symmetrical nor antisymmetrical, transitions directly induce current pulses in the electrical leads, a phenomenon called 'optical rectification'. In either case, the atoms are subjected to both a field at frequency ν_0 and a constant or slowly-varying field that performs an almost continuous measurement of the atomic state. This situation should be distinguished from the well-known Rabi evolution of atoms submitted to prescribed optical fields.

To be specific, let us assume that n_0 absorbing atoms are located between two conducting plates a distance d apart, small compared with wavelength. If E denotes the root-mean-square (rms) applied optical field, the rms voltage at frequency ν_0 between the plates is dE. For convenience, we denote by V this voltage divided by $(h\nu_0)^{1/2}$. The rules of quantum mechanics applied to the atom enable us to calculate the rms current at frequency ν_0 induced in the leads as a result of stimulated absorption. For convenience, I denotes the current divided by $(h\nu_0)^{1/2}$. For electric-dipole-allowed one-photon transitions, first-order perturbation shows that the quantum-mechanical expectation of I is proportional to V, i.e. the system is linear:

$$\langle I \rangle = GV \tag{1}$$

The optical conductance G, proportional to the number n_0 of atoms, can be considered constant since most atoms remain in the ground state. The atomic wavefunctions are assumed not to overlap; i.e. cooperative effects are not considered.

The rate Q of photons incident on the detector is defined as the incident electromagnetic power divided by the photon energy $h\nu_0$. According to previous definitions, Q is equal to VI. If both sides of Equation 1 are multiplied by V and we set $P \equiv V^2$, the average value of Q is proportional to P:

$$\langle Q \rangle = GP \tag{2}$$

Let us consider now instantaneous values of Q and P, and account for random jumps from the absorbing to the emitting state. We have

$$Q = GP + \mathbf{q} \qquad \mathbf{S}_{\mathbf{q}} = Q \tag{3}$$

A simple argument shows that the spectral density of the process $\mathbf{q}(t)$ is equal to the average rate $\langle Q \rangle$, denoted here simply as Q. The argument is that, in the special situation where V, and thus P, has a prescribed classical value, the atoms are independent of one another since they cannot 'communicate', so to speak; neither directly because their wavefunctions do not overlap, nor through the optical field since the latter has a prescribed value. Thus, atomic

transitions occur at independent times, i.e. they are Poisson-distributed. In one-photon processes each transition is caused by just one photon (according to the law of energy conservation). Thus, the fluctuation $\mathbf{q}(t)$ is also Poissonian, i.e. has a spectral density equal to the mean rate $\langle Q \rangle$.

In general P fluctuates but, well above threshold, the fluctuation ΔP is small. The fluctuation ΔQ of Q is given, according to Equation 3, by

$$\frac{\Delta Q}{Q} = \frac{\Delta P}{P} + \frac{\mathbf{q}}{Q} \qquad \mathbf{S}_{\mathbf{q}} + = Q \tag{4}$$

Because the relative variation of Q is small, the process remains approximately stationary. It is shown in Section 4 that amplitude fluctuations of high-power laser oscillators follow from Equation 4 alone.

Often, light is absorbed at the end of a possibly long (lossless, dispersionless) transmission line. If $D(t)$ denotes the absorbed photon rate and $Q(t)$ the rate at the laser output, it is intuitive that $D(t) = Q(t - \tau)$, where τ denotes the transit time along the line. Equation 3 is easily formulated in terms of propagating waves [4], instead of voltages and currents. According to the wave formalism, a weak B-wave, expressing the intrinsic fluctuations at the detector, and analogous to the vacuum wave considered in the quantum theory, propagates towards the laser cavity. This counterpropagating B-wave does not directly affect the detector current but it is instrumental in determining the amplitude fluctuations of the strong A-wave emanating from the laser cavity.

The emitter could also conceivably be connected to the laser cavity by means of a long transmission line. The formalism in [4] then shows that a weak counterpropagating B-wave propagates from the cavity to the emitter, and *anticipates* the intrinsic fluctuations at the emitter. It can be shown using the formalism of [4] that macroscopic causality is nevertheless preserved. Note that, according to our formalism, the optical field is classical, i.e. described by commuting functions of time, but not directly measurable. The electrons or atoms, in contradistinction, are quantized, and their number is measurable.

2.1. Relative intensity noise

Once the laser oscillator equations have been solved, ΔP is found to depend on $\mathbf{q}$ and on uncorrelated noise sources denoted collectively by z, according to

$$\frac{\Delta P}{P} = z + a\mathbf{q} \tag{5a}$$

$$\mathbf{S}_{\Delta P/P} = \mathbf{S}_z + aa^* Q \tag{5b}$$

where a is a constant that may be complex at nonzero frequencies, and stars denote complex conjugation. Using Equations 4 and 5, ΔQ and its spectral density read, respectively,

$$\frac{\Delta Q}{Q} = z + \left(a + \frac{1}{Q}\right)\mathbf{q} \tag{6}$$

$$\mathbf{S}_{\Delta Q/Q} = \mathbf{S}_z + \left(a + \frac{1}{Q}\right)\left(a + \frac{1}{Q}\right)^* Q = \mathbf{S}_{\Delta P/P} + \frac{1}{Q} + a + a^* \tag{7}$$

using Equation 5b.

Let us now introduce the relative-intensity-noise (RIN) defined as

$$\frac{\text{RIN}}{2} \equiv \mathbf{S}_{\Delta\mathbf{I}/\mathbf{I}} \equiv \mathbf{S}_{\Delta Q/Q} - \frac{1}{Q} = \mathbf{S}_{\Delta P/P} + a + a^* \tag{8}$$

the factor 2 being introduced because single-sided spectral densities are employed in optical engineering. The RIN (which may be negative) is useful because it is independent of linear losses, as shown in Section 3. Note that the so-called 'light-intensity' **I** is defined in Equation 8 by its spectral density, not explicitly as a function of time. It is employed in the phasor theory, Section 7.

Consider as an example high-power laser oscillators with quiet pumps, treated in Section 4. We have in that case $\Delta Q = 0$ at low frequencies, and

$$\frac{\Delta P}{P} = -\frac{\mathbf{q}}{Q} \tag{9}$$

implying, by comparison with Equation 5a that $z = 0$ and $a = -1/Q$. It then follows from Equations 5 and 8 that

$$\mathbf{S}_{\Delta P/P} = \frac{1}{Q} \qquad \frac{\text{RIN}}{2} = -\frac{1}{Q} \tag{10}$$

The difference between $\mathbf{S}_{\Delta P/P}$ and RIN/2 is obvious. The RIN is negative while the spectral density of ΔP is positive.

2.2. Optical cavities

When the n_0 absorbing atoms are located in a single-mode cavity modelled by a circuit with capacitance C tuned at frequency ν_0, it is convenient to introduce the photon number m and to exhibit the dependence of G on n_0 according to

$$\frac{G}{C} \equiv A_0 n_0 \qquad m \equiv CP \tag{11}$$

where A_0 is a constant, and it is assumed that all the atoms are subjected to the same optical field. Using that notation, Equation 3 reads

$$Q = A_0 n_0 m + \mathbf{q} \qquad \mathbf{S}_{\mathbf{q}} = Q \tag{12}$$

Since n_0 is presently a constant, the fluctuation ΔQ is written as

$$\Delta Q = A_0 n_0 \Delta m + \mathbf{q} \qquad \mathbf{S}_{\mathbf{q}} = Q \tag{13}$$

2.3. Emitters

Let us now turn our attention to emitters with n_a atoms in the absorbing state and $n_e > n_a$ atoms in the emitting state. If R_a and R_e denote the upward and downward rates respectively, the relations

$$R_a = An_a m + \mathbf{r}_a \qquad \mathbf{S}_{\mathbf{r}a} = R_a \tag{14a}$$

$$R_e = An_e m + \mathbf{r}_e \qquad \mathbf{S}_{\mathbf{r}e} = R_e \tag{14b}$$

are analogous to those in Equation 12. Note that same constant A applies to absorption and emission, and that the $\mathbf{r}_a$ and $\mathbf{r}_e$ noise sources are independent.

The net emission rate R is the difference of R_a and R_e,

$$R \equiv R_e - R_a = \mathbf{G}m + \mathbf{r} \qquad \mathbf{S}_r = R_a + R_e = (2n_p - 1)R \tag{15a}$$

where we have introduced the noise term **r**, the net optical gain **G**, and the population inversion factor n_p, according to

$$\mathbf{r} \equiv \mathbf{r}_e - \mathbf{r}_a \qquad \mathbf{G} \equiv A(n_e - n_a) \qquad n_p \equiv \frac{n_e}{n_e - n_a} \tag{15b}$$

The presently derived results, valid when the photon number m is large compared with unity, suffice to describe the noise properties of laser oscillators above threshold in the steady state. The relationship to single-atom behaviour is clarified below.

2.4. Quantum jumps

Let a single two-level atom, known to be in the ground state at $t = 0$, be submitted to a resonant prescribed optical field. The probability p that the atom will be found in the upper state at time $t = \Delta t$ is [7]

$$p = \sin^2(\Omega_R \, \Delta t/2) \approx (\Omega_R^2 \, \Delta t/4) \, \Delta t \tag{16}$$

if Δt is small compared with the Rabi period $2\pi/\Omega_R$. This probability is proportional to the optical field strength squared, say m. Because Δt is small, p may be viewed as the probability that a transition occurs during the time interval $t = 0$ to $t = \Delta t$. If imperfect continuous measurements are modelled by ideal measurements at times Δt, $2\Delta t, \ldots,$ a telegraphic (Markov) process is generated: once the atom is in the upper state, there is a probability p that a downward transition will occur during any subsequent time interval Δt, and so on. Under the conditions just specified, it makes sense to assert that the atom is, at any instant, either in the ground state or in the upper state. In the case of a detector, electrons that have reached the upper state quickly decay to the ground state, e.g. by flowing through external electrical leads. Accordingly, the detected signal fluctuates at the shot-noise level, even for a single atom under some conditions.

2.5. Spontaneous emission

The refined formulation given below that includes spontaneous decay would be required to describe below-threshold behaviour (possibly using numerical methods) and laser start-up. The probability for downward transition of emitting-state atoms is not quite proportional to m as asserted earlier. More accurately,

$$p = A(m + 1) \tag{17a}$$

where A is a constant. The term '1' in Equation 17, independent of the number of photons already existing in the mode, follows from linear noise theory [6]. If this term is considered Equation 15a reads

$$R = \mathbf{G}(m + n_p) + \mathbf{r} \qquad \mathbf{S}_{\mathbf{r}} = (2n_p - 1)R \tag{17b}$$

For an oscillator with cavity photon-lifetime τ_p, steady-state oscillation requires that $\langle \mathbf{G} \rangle = 1/\tau_p$, and thus the additional term in the expression for R is $\mathbf{G}n_p \approx n_p/\tau_p$.

This term is viewed as the spontaneous emission rate in the mode. It is often evaluated through a complicated procedure. One first employs the relation established by Einstein between the total spontaneous emission rate S_r and the gain $\langle \mathbf{G} \rangle$, and considers that only a fraction $\beta \approx 10^{-5}$ of this rate is effective as a noise source, where β is the product of two terms:

one obtained by considering that the oscillator filters one spatial transverse mode out of many, the other being the ratio of the frequency spacing between adjacent longitudinal modes and the spontaneous-emission spectral width. This procedure may be useful when spontaneous carrier-recombination is essentially radiative, since the total spontaneous-emission rate is then known from the threshold current. Its usefulness may be questioned for 1.55 μm laser diodes because carrier recombination is mostly nonradiative in that case ($S_r \ll S$).

Let us emphasize that the value of β is fixed by general physical principles, as discussed above, and that β should not be considered an adjustable, empirically determined, parameter. This is why, in particular, the strong observed relaxation-oscillation damping of laser diodes cannot be explained by postulating large β-factors. Gain compression (to be discussed later) is required to explain the observed effect.

Measurements of the β-spatial-factor with the detector placed in front of the laser-diode edge suffer from inadequate spatial filtering and are unreliable. A crude spatial-mode filtering consisting of a pair of apertures appropriately spaced apart (with a Fresnel number of the order of unity) should be employed. Otherwise spontaneous emission rays may enter the detector and lead to an overestimation of the β-value.

3. Role of optical losses

Let Q denote as before the total photon emission rate. The detected rate D may be less than Q for various reasons. For example, the detector might collect only part of the light emitted by the source, as shown in Fig. 1, or a beam splitter might divert part of the emitted light. In any event the attenuator is assumed linear, cold and frequency-insensitive.

For laser diodes without gain compression

$$\frac{D}{Q} = \frac{I_d}{I_e - I_{th}} \tag{18}$$

where I_d, I_e, I_{th} denote respectively the detected, injected, and threshold currents.

To model the absorption mechanism, let the conductance G previously considered be split

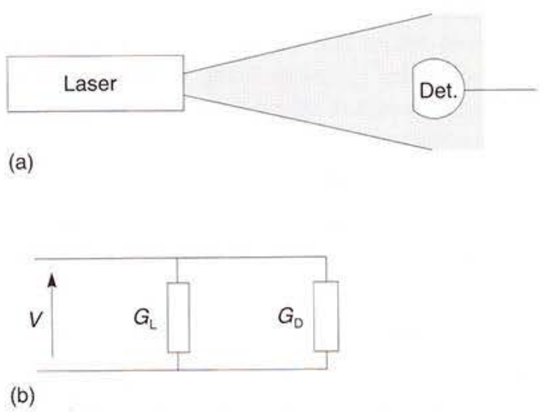

Figure 1 Optical losses. (a) Light from the laser is only partly collected by the detector. If the output photon rate is Q, the detected photon rate is $D < Q$. (b) Equivalent circuit. The total conductance G is split into a loss-conductance G_L and a detector conductance G_D. In the configuration in (a) the conductance G_L represents an absorber presumed to exist at infinity.

into two constant conductances, G_L and G_D, with

$$G_L + G_D = G \tag{19}$$

G_L being arbitrarily elected to describe the loss while G_D represents the detector. If both sides of Equation 19 are multiplied by $\langle P\rangle$, where $P \equiv V^2$, the average rates obey

$$\langle L\rangle + \langle D\rangle = \langle Q\rangle \qquad \langle L\rangle = G_L\langle P\rangle \qquad \langle D\rangle = G_D\langle P\rangle \tag{20}$$

Going back to fluctuations, the basic equation (4) splits into

$$\Delta L = G_L\,\Delta P + \ell \qquad S_\ell = L \tag{21}$$

and

$$\Delta D = G_D\,\Delta P + \mathbf{d} \qquad \mathbf{S}_\mathbf{d} = D \tag{22}$$

ΔL and ΔD represent the fluctuations of the photon rates that are absorbed in the loss and detection conductances, respectively. ℓ and $\mathbf{d}$ are independent fluctuations that add up to $\mathbf{q}$:

$$\ell + d = q \tag{23}$$

3.1. Relative intensity noise

Equation 22 can be written

$$\frac{\Delta D}{D} = \frac{\Delta P}{P} + \frac{\mathrm{d}}{D} \tag{24}$$

because $G_D\langle P\rangle = \langle D\rangle$. According to Equation 5a, $\Delta P/P$ may be written in the form

$$\frac{\Delta P}{P} = z + a\mathbf{d} + a\ell \tag{25a}$$

$$\mathbf{S}_{\Delta P/P} = \mathbf{S}_z + aa^*D + aa^*L \tag{25b}$$

if Equation 23 is used, since the three terms in Equation 25a are uncorrelated and the spectral densities of $\mathbf{d}$ and ℓ are respectively equal to D and L.

We have from Equations 24 and 25

$$\frac{\Delta D}{D} = z + \left(a + \frac{1}{D}\right)\mathbf{d} + a\ell \tag{26a}$$

$$\mathbf{S}_{\Delta D/D} = \mathbf{S}_z + \left(a + \frac{1}{D}\right)\left(a^* + \frac{1}{D}\right)D + aa^*L$$

$$= \mathbf{S}_{\Delta P/P} + \frac{1}{D} + a + a^* \tag{26b}$$

if Equation 25b is employed.

$\Delta P/P$ and a being independent of the attenuation, it follows from Equation 26 that the RIN defined in Equation 7,

$$\frac{\mathrm{RIN}}{2} \equiv \mathbf{S}_{\Delta Q/Q} - \frac{1}{Q} = \mathbf{S}_{\Delta D/D} - \frac{1}{D} \tag{27}$$

is independent of the attenuation and is therefore a useful concept. However, 'light intensity' itself is not a directly measurable quantity.

The RIN is best measured with a pair of low-noise (preferably cooled) p-i-n photodiodes (Hanburry-Brown and Twiss-type experiment). The average currents give the incident power. When the radiofrequency outputs are multiplied together to eliminate the uncorrelated parts of the photocurrents, the RIN is directly obtained. The accuracy is best if the optical attenuation between the laser and the detector is kept as small as possible, i.e. if the detector current is close to the laser diode current, but one must also ensure that no light is reflected back to the laser. Unfortunately, existing optical isolators introduce losses and spurious reflections. Note also that the measurements should be performed above, approximately, 1 MHz to avoid the occurrence of $1/f$ noise and (crystal) thermal effects.

For calibration, a light source of equal power fluctuating at the shot-noise level is needed. Thermal sources and light-emitting-diodes (LED) are, in principle, super-Poissonian. But they may be employed in the guise of Poissonian sources when the detector bandwidth is very small compared with the optical bandwidth.

3.2. Correlation

Correlations between various measurable fluctuations are important to better understand the origin of noise, and also in applications. In the case of laser diodes we have rather easy access to the electron number fluctuation through the electrical voltage across the diode. This noise term may be employed to minimize amplitude or phase noise.

A quantity x, independent of the attenuation, such as atomic number fluctuation, can be set equal to $y + b\mathbf{q}$, where b is a constant, and y and $\mathbf{q}$ are uncorrelated. Since the cross-spectral density of $\mathbf{q}$ and d/D is unity, the relation

$$\mathbf{S}_{x,\Delta Q/Q} = \mathbf{S}_{x,\Delta D/D} \tag{28}$$

follows from Equation 24. Note that it is here essential to consider $\Delta D/D$ and not ΔD itself.

It is therefore appropriate to define attenuation-independent correlations

$$\mathbf{C}_{\mathrm{nd}} \equiv \frac{\mathbf{S}_{\Delta n,\Delta D}}{[\mathbf{S}_{\Delta n}|\mathbf{S}_{\Delta D} - D|]^{1/2}} = \frac{\mathbf{S}_{\Delta n/n,\Delta D/D}}{[\mathbf{S}_{\Delta n/n}|\mathrm{RIN}/2|]^{1/2}} \tag{29}$$

where we have taken for x the atom-number fluctuation.

For laser diodes, Δn in Equation 29 is proportional to the fluctuation ΔU of the electrical voltage across the intrinsic diode. At high power, the fluctuation ΔU may become negligible compared with the Nyquist noise of the diode series resistance, of the order of 1 Ω. But since this thermal noise is independent of the other noise sources it can, in principle, be subtracted out. Note that, above threshold, the series resistance is almost equal to the measured diode electrical impedance.

For measuring electrical–optical correlation, it is convenient to introduce a long delay line, either optical between the laser and the detector or, equivalently, electrical between the detector and the correlator as done in [27]. As the frequency varies, the relative phase of Δn and ΔD varies quickly and calibration is easier.

To measure the photon number m at one time it suffices in principle to disconnect the optical cavity from the active material and count the number of photoelectrons in the absorber (photocount) over a period of time large compared with the cavity decay time τ_{p}. The cross-correlation $\langle \Delta m(\tau)\ \Delta n(0)\rangle$, $\tau > 0$, may then be obtained from measurements performed on the laser itself.

The direct continuous measurement of the photon number m has been shown in recent years

to be possible with the help of Kerr's effect (classically described as an increase of the refractive-index as a function of light intensity). Changes of the m-value result in measurable changes of the phase of a probe-wave phase whose frequency is different from the signal frequency. Such a measurement would not affect the measured m-value (quantum-nondemolition detection), but it would scramble the oscillator phase and deeply affect the laser operation. Thus, it appears that the photon number cannot be continuously and directly monitored.

The spectral density of m can be determined, however, if the detected rate $\Delta Q(t)$ is available, as shown in Section 4. Because m is measurable, albeit indirectly, its spectral density must be positive (see the appendix).

4. Laser oscillators neglecting spontaneous decay

The oscillator model consists of a single-cavity resonating at frequency ν_0 and containing atoms in the ground state playing the role of detector. The absorbing atoms may be located at the end of a long transmission line and be identified with a detector. It is then assumed that no light is reflected back to the laser cavity. Whether the absorbing atoms are located inside or outside the cavity turns out to be immaterial.

Spontaneous decay from emitting to absorbing states is considered negligible. This condition is well fulfilled for some laser diodes that operate at, say, ten times the threshold current. In such circumstances spontaneous emission is entirely irrelevant.

The fluctuation ΔQ of the rate at which photons are absorbed is related to the fluctuation Δm of the number m of photons in the cavity according to Equation 13:

$$\Delta Q = \frac{\Delta m}{\tau_{\mathrm{p}}} + \mathbf{q} \qquad \mathbf{S}_{\mathbf{q}} = Q \tag{30}$$

where we have introduced the photon lifetime $1/\tau_{\mathrm{p}} \equiv A_0 n_0$.

To offset the loss, three-level atoms (with an emitting level e, an absorbing level a and a ground-state level g) are pumped at rate J, as shown in Fig. 2. It is assumed that electrons

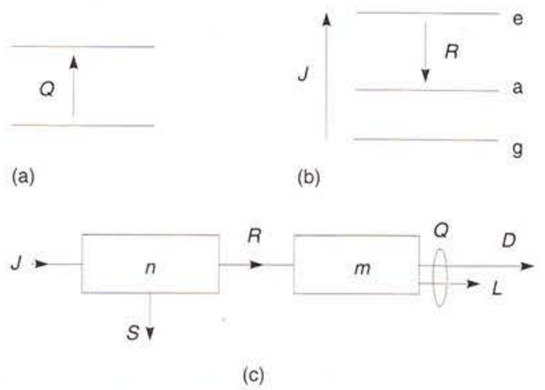

Figure 2 (a) Schematic representation of a two-level atom in the absorbing state. Q denotes the rate at which incident photons are absorbed. (b) Three-level atom with working levels a (absorbing level) and e (emitting level). Electrons are pumped from the ground level g to the emitting level at rate J, which may be nonfluctuating ($\xi = 0$). Decay from the absorbing state a to the ground state is assumed in this paper to be instantaneous, so that there are no atoms in the absorbing level. R denotes the stimulated emission of radiation rate. Spontaneous decay is not shown. (c) This schematic represents the particle rates (J, S, R, L and D, with $L + D = Q$), and particle numbers: atom number n and photon number m. S represents the spontaneous (radiative or nonradiative) decay.

in the absorbing level decay almost instantaneously to the ground state by some process and become optically inactive. Accordingly, the number n_a of atoms in the absorbing state is negligible: $n_a \approx 0$. Stimulated absorption does not occur and we need only consider the number $n_a \equiv n$ of atoms in the emitting state.

The rate at which atoms in the emitting state accumulate is the difference between the pumping rate J and the rate R of decay,

$$\frac{dn}{dt} = J - R \tag{31}$$

On the other hand, the cavity equation

$$\frac{dm}{dt} = R - Q \tag{32}$$

asserts that the rate dm/dt at which photons accumulate in the cavity is the difference between the rate R at which photons enter the cavity and the rate Q at which they leave the cavity.

If we consider small-amplitude fluctuations at baseband frequency $f \equiv \Omega/2\pi$, we may replace d/dt by $i\Omega$ and Equations 31 and 32 read, respectively,

$$i\Omega\,\Delta n = \Delta J - \Delta R \tag{33}$$

$$i\Omega\,\Delta m = \Delta R - \Delta Q \tag{34}$$

Obviously, the number of photocounts is the variation of $n + m$ over some time interval; for quiet pumps $\Delta J = 0$. Accordingly, $m(t)$ can be obtained experimentally from measurements of $n(t)$ and $Q(t)$. But $m(t)$ cannot be measured directly without scrambling the phase.

4.1. High power

To best exhibit the essential features, assume that the photon number m is much larger than the atom number. Atoms injected in the cavity then decay almost instantaneously. Because J and R are large, the term dn/dt can be neglected and thus $\Delta R = \Delta J$, where ΔJ represents the pump fluctuation, which is independent of the other noise sources. Equations 30 and 34 read

$$i\Omega\,\Delta m = \Delta J - \Delta Q = \Delta J - \frac{\Delta m}{\tau_p} - \mathbf{q} \qquad \mathbf{S}_{\mathbf{q}} = Q \tag{35}$$

a relation that can be solved for Δm:

$$\Delta m = \frac{\Delta J - \mathbf{q}}{\tau_p^{-1} + i\Omega} \tag{36}$$

The spectral density of Δm is thus

$$\mathbf{S}_{\Delta m} = \frac{\mathbf{S}_{\Delta J} + Q}{\tau_p^{-2} + \Omega^2} \tag{37}$$

Let us now set

$$\mathbf{S}_{\Delta J} = \xi J \tag{38}$$

where $\xi = 0$ for a quiet pump and $\xi = 1$ for a Poissonian pump. Equation 37 reads

$$Q\mathbf{S}_{\Delta m/m} = \frac{1+\xi}{1+\Omega^{\circ 2}} \qquad \Omega^{\circ} \equiv \Omega\tau_p \equiv \frac{f}{f_0} \tag{39}$$

where we have used the relation $\langle Q \rangle = \langle m \rangle / \tau_p$.

The variance $\langle \Delta m^2 \rangle$ of m is the integral of its spectral density over frequency f from minus to plus infinity. We readily obtain from Equation 39

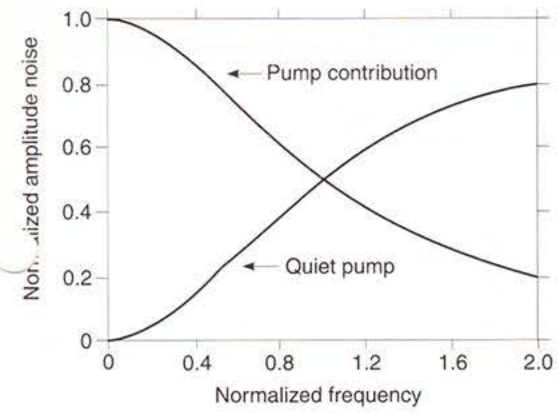

Figure 3 Variation of $\chi \equiv Q^{-1}\,\mathbf{S}_{\Delta Q}$ (spectrum of detected photoelectron rate relative to the average rate Q) as a function of f/f_0 (ratio of baseband frequency f and the cold-cavity linewidth f_0). One curve gives the pump fluctuation contribution for $\xi = 1$. The other curve is the quiet-pump contribution. The sum of the two contributions is at the shot-noise level at any frequency, i.e. $\chi = 1$.

$$\langle \Delta m^2 \rangle = (1 + \xi)\frac{m}{2} \tag{40}$$

Thus, for regular pumping ($\xi = 0$) the photon number variance is half the mean, while for Poissonian pumping the photon number variance equals the mean.

A quantity of greater interest is the detected photon rate Q. Inserting the result for Δm in Equation 36 into Equation 30, we obtain

$$\Delta Q = \frac{\Delta m}{\tau_p} + \mathbf{q} = \frac{\Delta J + i\Omega^{\circ}\mathbf{q}}{1 + i\Omega^{\circ}} \tag{41}$$

Equation 41 shows that at small frequencies ($\Omega^{\circ} = 0$) photon rate and pump fluctuations coincide: $\Delta Q = \Delta J$. In particular, the photon rate does not fluctuate for a quiet pump ($\Delta J = 0$). This fundamental conclusion, which escaped the attention of early laser-noise theorists, was discovered by Golubev and Sokolov [16] from quantum theory. It is remarkable that it follows from an elementary classical theory as well.

According to Equation 41,

$$\chi \equiv Q^{-1}\mathbf{S}_{\Delta Q} = \frac{\xi}{1+\Omega^{\circ 2}} + \frac{\Omega^{\circ 2}}{1+\Omega^{\circ 2}} \tag{42}$$

which is represented in Fig. 3. The first term on the right-hand side of Equation 42, which expresses the laser response to pump fluctuations, decays as a function of frequency. The second term vanishes at zero frequency and increases with frequency to reach the shot-noise level (SNL) at large frequencies. The sum of the two terms turns out to be at the SNL (i.e. $\chi = 1$) at any frequency.

From Equation 42 the RIN defined in Equation 8 reads

$$Q\,\frac{\mathrm{RIN}}{2} \equiv Q\mathbf{S}_{\Delta Q/Q} - 1 = \frac{\xi - 1}{1+\Omega^{\circ 2}} \tag{43}$$

Within the present high-power approximation, and for $\xi = 1$, the correlation between atomic number and detected current fluctuations vanishes at all frequencies [3].

4.2. Carrier dynamics

When the atomic number dynamics is considered, a large relaxation-oscillation peak appears in the noise spectrum. To see this important fact, let us restore the dn/dt term. Since $n_a \approx 0$, full population inversion is assumed in the expression of R in terms of m, (Equation 15), i.e. $n_p = 1$.

The steady-state condition is

$$\langle J \rangle = \langle R \rangle = \langle Q \rangle = A\langle n \rangle \langle m \rangle = \frac{\langle m \rangle}{\tau_p} \tag{44}$$

The rate equations in Equations 33 and 34 read

$$i\Omega^\circ \epsilon \frac{\Delta n}{n} = \frac{\Delta J}{Q} - \frac{\Delta R}{Q} \tag{45a}$$

$$i\Omega^\circ \frac{\Delta m}{m} = \frac{\Delta R}{Q} - \frac{\Delta Q}{Q} \tag{45b}$$

where $\Omega^\circ \equiv \Omega\tau_p$, $\epsilon \equiv n/m$, and

$$\frac{\Delta R}{Q} = \frac{\Delta n}{n} + \frac{\Delta m}{m} + \frac{\mathbf{r}}{Q} \qquad \mathbf{S}_r = R = Q \tag{45c}$$

$$\frac{\Delta Q}{Q} = \frac{\Delta m}{m} + \frac{\mathbf{q}}{Q} \qquad \mathbf{S}_q = Q \tag{45d}$$

Solving Equation 45 for Δn, Δm and ΔQ, we obtain,

$$Qf(\epsilon, \Omega^\circ)\frac{\Delta n}{n} = i\Omega^\circ \Delta J - (1 + i\Omega^\circ)\mathbf{r} + \mathbf{q} \tag{46a}$$

$$Qf(\epsilon, \Omega^\circ)\frac{\Delta m}{m} = \Delta J + i\Omega^\circ \epsilon \mathbf{r} - (1 + i\Omega^\circ \epsilon)\mathbf{q} \tag{46b}$$

$$Qf(\epsilon, \Omega^\circ)\frac{\Delta Q}{Q} = \Delta J + i\Omega^\circ \epsilon \mathbf{r} + [i\Omega^\circ(1-\epsilon) - \epsilon\Omega^{\circ 2}]\mathbf{q} \tag{46c}$$

where

$$f(\epsilon, \Omega^\circ) \equiv 1 - \epsilon\Omega^{\circ 2} + i\Omega^\circ \tag{46d}$$

The spectral densities of Δn, Δm and ΔQ (expressed in terms of the RIN) are thus, respectively,

$$Q\mathbf{S}_{\Delta n/n} = \frac{2 + (\xi + 1)\Omega^{\circ 2}}{F(\epsilon, \Omega^\circ)} \tag{47a}$$

$$Q\mathbf{S}_{\Delta m/m} = \frac{\xi + 1 + 2\epsilon^2\Omega^{\circ 2}}{F(\epsilon, \Omega^\circ)} \tag{47b}$$

$$Q\frac{\text{RIN}}{2} = \frac{\xi - 1 + 2\epsilon^2\Omega^{\circ 2}}{F(\epsilon, \Omega^\circ)} \tag{47c}$$

where

$$F(\epsilon, \Omega^\circ) \equiv |f(\epsilon, \Omega^\circ)|^2 = (1 - \epsilon\Omega^{\circ 2})^2 + \Omega^{\circ 2} \tag{47d}$$

The zero of $F(\Omega^\circ)$ in the complex plane gives relaxation-oscillation frequency and damping. The relaxation frequency f_r is given approximately ($\epsilon \equiv n/m \gg 1$) by $\epsilon\Omega^{\circ 2} \approx 1$, i.e.

$$f_r \approx f_0\sqrt{m/n} \tag{48}$$

Let us recall that $\xi = 1$ for a Poissonian pump and $\xi = 0$ for a quiet pump.

4.3. Atom and photon number variance

The photon number variance $\langle \Delta m^2 \rangle$ most often considered in early laser-noise theories is obtained by integrating over frequency the spectral density given in Equation 47b.

We will need the mathematical results

$$\int_{-\infty}^{+\infty} \frac{dx}{F(\epsilon, x)} = \int_{-\infty}^{+\infty} dx \frac{\epsilon x^2}{F(\epsilon, x)} = \pi \qquad F(\epsilon, x) \equiv (1 - \epsilon x^2)^2 + x^2 \tag{49}$$

that hold for any ϵ-value. The first integral is easily evaluated for $\epsilon = 0$. To show that this integral does not depend on ϵ, it suffices to differentiate with respect to ϵ. Two terms are found that cancel out when the change of variable $\epsilon x = 1/y$ is made in the second one. The same change of variable shows that the second integral is equal to the first.

Using Equation 49, Equations 47a and b lead to

$$\frac{\langle \Delta m^2 \rangle}{m} = \frac{\langle \Delta n^2 \rangle}{n} = \frac{n}{m} + \frac{\xi + 1}{2} \tag{50}$$

The expression in Equation 40 is recovered from Equation 50 when n is neglected in comparison with m, i.e. at high powers.

4.4. Langevin's form

For the sake of comparison with previous theories, let us lump together the independent noise terms and write Equation 45 in the form

$$i\Omega\,\Delta n = -\frac{\Delta n}{\epsilon\tau_p} - \frac{\Delta m}{\tau_p} + F_n \tag{51a}$$

$$i\Omega\,\Delta m = \frac{\Delta n}{\epsilon\tau_p} + F_m \tag{51b}$$

where Langevin's forces are defined as

$$F_n \equiv \Delta J - \mathbf{r} \qquad F_m \equiv \mathbf{r} - \mathbf{q} \tag{51c}$$

The (proper and cross) spectral densities of F_n and F_m follow from the above expressions in terms of $\mathbf{r}$ and $\mathbf{q}$ and the spectral densities given earlier for these quantities, namely (introducing for generality the population inversion factor n_p, which was unity in previous equations)

$$\mathbf{S}_n = (2n_p - 1 + \xi)Q \qquad \mathbf{S}_m = 2n_pQ \qquad \mathbf{S}_{nm} = (1 - 2n_p)Q \tag{51d}$$

The spectral density of Δm reads, with that notation,

$$F(\epsilon, \Omega^\circ)\tau_p^{-2}\mathbf{S}_{\Delta m} = \mathbf{S}_n + (1 + \epsilon^2\Omega^{\circ 2})\mathbf{S}_m + 2\mathbf{S}_{nm} \tag{51e}$$

which, of course, gives Equation 47b again when Equation 51d is employed and $n_p = 1$. We note that at low frequencies the value of n_p does not affect the photon number or photon rate spectral densities.

4.5. Equivalent electrical circuit

It is very useful for classroom demonstration and simulation in systems to express the above results in terms of an equivalent electrical circuit. The atom–photon dynamics is usually modelled by a resonating inductance–capacitance circuit. In order to preserve the symmetry between emitting and absorbing elements, it is preferable to consider instead negative and positive capacitances. Negative elements have been realized with active electronics (negative-impedance converters) [28].

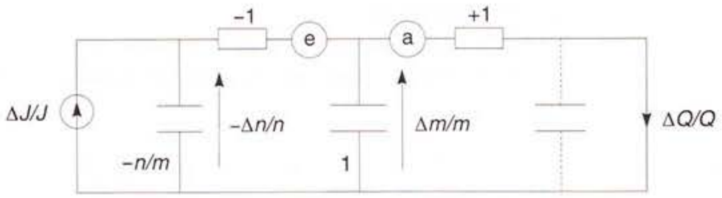

Figure 4 Equivalent electrical circuit for a laser oscillator with spontaneous decay neglected. For simplicity the photon lifetime is taken as the time unit. The cavity is modelled by a capacitance equal to the photon lifetime τ_p, i.e. unity. The emitting atoms are modelled by a resistance equal to -1, in series with a voltage source whose spectral density is $1/Q$, and a parallel capacitance opposite to the ratio n/m of electron and photon numbers. The absorber (detector) is modelled by a resistance of unity in series with a voltage-noise source of spectral density equal to $1/Q$. The detector capacitance (carrier storage and parasitic) is shown in dotted lines only to emphasize the symmetry between emitting and absorbing elements. It is not considered in detail. The current source on the left ($\Delta J/J$) represents the pump modulation and noise, while the current $\Delta Q/Q$ on the right represents the relative detected-current modulation and noise.

The equivalent circuit shown in Fig. 4 follows from Equation 45 written in the form

$$\frac{\Delta m}{m} = \frac{\Delta Q}{Q} - \frac{\mathbf{q}}{Q} \tag{52a}$$

$$\frac{\Delta m}{m} = -\frac{\Delta n}{n} - \frac{\mathbf{r}}{Q} + \frac{\Delta J}{J} - \mathrm{i}\Omega^{\circ}\frac{n}{m}\frac{\Delta n}{n} \tag{52b}$$

The voltage across the middle capacitance of value unity (representing the optical cavity) is $\Delta m/m$. The current $\Delta Q/Q$ flowing through a unity resistance gives the the first term on the right-hand side of Equation 52a, while the second term is a voltage noise source denoted 'r' in Fig. 4 with spectral density $1/Q$. The dotted-line capacitor corresponding to carrier storage in the detector is shown only for the sake of symmetry.

Looking now at the left-hand side of the schematic, we note that $\Delta m/m$ in Equation 52b is the sum of three terms: (1) the voltage $-\Delta n/n$ across the capacitor expressing carrier storage in the laser active material, (2) the voltage noise source 'e' of spectral density $1/Q$, and (3) the voltage across the resistance of value -1. The current flowing through this resistance is indeed the injected pump fluctuation $\Delta J/J$ and the current flowing through the capacitor of value $-n/m$, subjected to the voltage $-\Delta n/n$, i.e. $\mathrm{i}\Omega^{\circ}\,\Delta n/m$.

This schematic makes it obvious that at low frequencies $\Delta Q = \Delta J$. Since the capacitances can be removed at low frequencies, the current source on the left in Fig. 4 flows unimpeded into the right lead, and thus $\Delta J/J = \Delta Q/Q$. The relation $\Delta J = \Delta Q$ follows since the average rates J and Q are equal.

The equivalent circuit can easily be generalized. A nonzero driver admittance is represented by an element connected in parallel with the current source shown on the left. To account for spontaneous carrier recombination one must also introduce a noise-current source. Spectral-hole burning leads to nonunity values for the resistance on the left of the schematic, the noise sources being unaffected.

5. Laser oscillators with spontaneous decay

We now relax the assumption made in the previous section that spontaneous decay is negligible

in comparison with stimulated decay. Equations 31 and 32 then read

$$\frac{\mathrm{d}n}{\mathrm{d}t} = J - S - R \tag{53a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = R - Q \tag{53b}$$

The steady-state condition obtained by setting $\mathrm{d}n/\mathrm{d}t = \mathrm{d}m/\mathrm{d}t = 0$ is

$$\langle J\rangle - \langle S\rangle = \langle R\rangle = \langle Q\rangle \tag{54}$$

Radiative spontaneous decay is proportional to the number n of atoms in the emitting state and fluctuates at the SNL,

$$S = \frac{n}{\tau_s} + \mathbf{s} \qquad \mathbf{S}_{\mathbf{s}} = S \tag{55a}$$

where τ_s is the spontaneous-recombination lifetime. Equation 55a reads, to first order,

$$\Delta S = \frac{\Delta n}{\tau_s} + \mathbf{s} \qquad \mathbf{S}_{\mathbf{s}} = S = J - Q \tag{55b}$$

The rate R at which photons are emitted (Equations 15 and 16 with $n_e \equiv n$, $n_a \approx 0$, $n_p = 1$) reads as before

$$R = Anm + \mathbf{r} \qquad \mathbf{S}_{\mathbf{r}} = R \tag{56}$$

i.e. to first order

$$\Delta R = Am\,\Delta n + An\,\Delta m + \mathbf{r} \qquad \mathbf{S}_{\mathbf{r}} = R = Q \tag{57}$$

Equation 53 with Equations 55 and 57 constitute the full set of rate equations.

For simplicity, the fluctuations will be evaluated at zero frequency. Setting $\mathrm{d}/\mathrm{d}t = 0$ in Equation 53 we have

$$\Delta J - \Delta S = \Delta R = \Delta Q \tag{58}$$

that is, explicitly, using Equations 55b, 57 and 4,

$$\Delta J - \frac{\Delta n}{\tau_s} - \mathbf{s} = Am\,\Delta n + An\,\Delta m + \mathbf{r} = \frac{\Delta m}{\tau_p} + \mathbf{q} \tag{59}$$

Because $A\langle n\rangle = 1/\tau_p$, the last equality in Equation 59 reads simply

$$Am\,\Delta n = \mathbf{q} - \mathbf{r} \tag{60}$$

Since $\mathbf{q}$ and $\mathbf{r}$ are independent and have both spectral densities equal to Q, and $A\langle n\rangle\langle m\rangle = \langle Q\rangle$, the spectral density of $\Delta n/n$ is

$$\mathbf{S}_{\Delta n/n} = \frac{2}{Q} \tag{61}$$

When the expression for Δn in Equation 60 is introduced into the first term in Equation 59, we obtain

$$\Delta Q = \Delta J - \mathbf{s} - \frac{S}{Q}(\mathbf{q} - \mathbf{r}) \tag{62}$$

where the relation $A\langle m\rangle\tau_s = \langle Q\rangle/\langle S\rangle$ is used. The noise terms ΔJ, $\mathbf{s}$, $\mathbf{q}$ and $\mathbf{r}$ in Equation 62 are independent and have spectral densities

$$\mathbf{S}_{\Delta J} = \xi J \qquad \mathbf{S_s} = S \qquad \mathbf{S_q} = \mathbf{S_r} = Q \tag{63}$$

where $\xi = 0$ for a quiet pump and $\xi = 1$ for a Poissonian pump. Thus

$$\mathbf{S}_{\Delta Q} = \xi J + S + 2\left(\frac{S}{Q}\right)^2 Q \tag{64}$$

The RIN defined in Equation 7 may be written as

$$S\,\frac{\mathrm{RIN}}{2} = S\left(\mathbf{S}_{\Delta Q/Q} - \frac{1}{Q}\right) \tag{65}$$

Thus, according to Equation 64

$$S\,\frac{\mathrm{RIN}}{2} = \frac{S}{Q}\left[\xi\left(\frac{J}{Q}\right) + \frac{S}{Q} + 2\left(\frac{S}{Q}\right)^2 - 1\right] = [2 - r(1-\xi)]\,\frac{1+r}{r^3} \tag{66}$$

where

$$r \equiv \frac{J}{S} - 1 = \frac{Q}{S} \tag{67}$$

is the ratio of injected to threshold pump rates minus 1. On the left-hand side of Equation 66 the RIN has been normalized by S rather than Q as in the previous section because we now wish to investigate the effect of pumping rate on intensity noise. For a quiet pump ($\xi = 0$) the RIN is negative when $r > 2$. For Poissonian pumps ($\xi = 1$) the RIN, equal to $4JS^2/Q^3$, is inversely proportional to the square of the output power much above threshold. The above expressions for the RIN can also be obtained by specializing the expressions in [3]. The variations of the RIN given in Equation 66 as a function of the injected-to-threshold pump rate ratio are shown in Fig. 5 for $\xi = 0$.

If the threshold current of a laser diode is 1 mA and the wavelength is 1.55 μm, for example, the RIN at four times threshold is 74×10^{-18} for a pump fluctuating at the SNL, and -37×10^{-18} for a quiet pump. Measured values are generally much higher as a result of spurious-mode oscillation.

The attenuation-independent correlation $\mathbf{C}_{nd}$ between the fluctuations of n (or of the electrical voltage U across the intrinsic diode for a laser diode) and the detected current fluctuations defined in Equation 29 reads

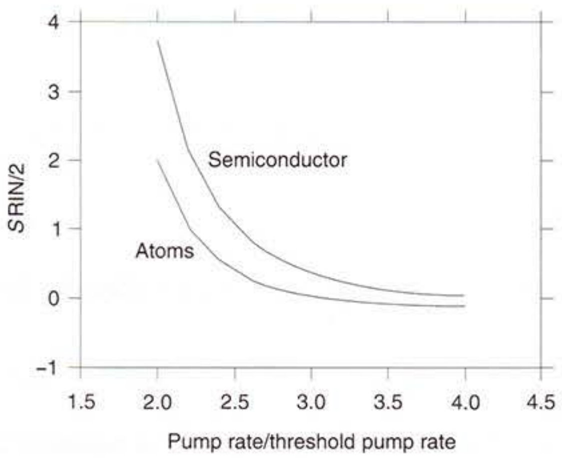

Figure 5 Variation of SRIN/2 (where $S = I_{th}$ denotes the spontaneous decay rate, and RIN the low-frequency relative-intensity noise) as a function of $r + 1 \equiv I_e/I_{th}$, where I_e is the pump rate and I_{th} the threshold pump rate. The lower curve is for three-level atoms with quiet pumps ($\xi = 0$). The upper curve is for a laser-diode (see Section 6) with $\xi = 0$, $\zeta = 1$ (radiative spontaneous decay) and a, defined in Equation 72b, equal to 2.35.

$$|\mathbf{C}_{nd}|^{-2} = |2 - r(1-\xi)|\,\frac{1+r}{2} \tag{68a}$$

The correlation tends to zero at large powers, in agreement with the result given at the end of the previous section. It is singular at $r = 2/(1-\xi)$ since the RIN then vanishes. In the special se where $\xi = 1$ (Poissonian pump) we have simply

$$\mathbf{C}_{nd} = -\sqrt{I_{th}/I_e} \tag{68b}$$

where I_{th} and I_e are, respectively, the threshold and injected currents.

In contrast to $\mathbf{C}_{nd}$, which is attenuation independent, the normalized cross-spectral density between the atom electron number n and the detected rate D (or, equivalently, between the electrical voltage U across the intrinsic diode junction and the detected current I_d): $\mathbf{S}_{\Delta n\,\Delta D}\,[\mathbf{S}_{\Delta n}\mathbf{S}_{\Delta D}]^{-1/2}$, sometimes called 'coherence' and denoted by γ, depends strongly on the optical attenuation. It is easily shown from previous equations that γ drops to negligible values just above threshold when the optical attenuation between laser and detector is large (say, 20 dBs).

This fact is not always appreciated for the following reason. Some authors assume that the detector rate D is at any instant proportional to the photon number m, $D = \sigma m/\tau_p$, where the constant $\sigma < 1$ expresses the optical loss. They further identify RIN/2 as the spectral density of $\Delta D/D$. Since both ΔD and D are multiplied by the same σ-value, the RIN/2 evaluated in that way seems to be attenuation independent. But this quick proof is erroneous, because it ignores the McCumber shot-noise term in the detection process and the random photon decimation caused by optical losses. When the same principles are applied to the coherence γ, one is tempted to conclude that γ is attenuation independent because, again, the same σ-value pears in the numerator and denominator of the expression defining γ. But, as we have en, such a conclusion may be grossly erroneous.

Finally, let us recall that, below threshold, the laser output is thermal-like (linearly amplified spontaneous emission) and that the shot-noise theory is wholly inapplicable (as McCumber, of course, was well aware).

6. Laser diodes

It follows from the Pauli principle that one electron at most can occupy a given state. Electron energies therefore must spread out, even at $T = 0$ K. Only a few per cent of the electron–hole pairs (say, n) have an energy spacing appropriate for interaction with the optical field. The total number of electrons, equal to the total number of holes, will be denoted $N \gg n$. In the standard theory of laser diodes it is assumed that the electrons are in thermal equilibrium in the conduction band, and the holes are in thermal equilibrium in the valence band, with temperatures equal to that of the crystal. This assumption allows quasi-Fermi levels in the valence and conduction bands to be defined, with an energy spacing equal to the electrical voltage applied to the intrinsic diode multiplied by the electron charge (see, e.g., [29]). This theory is applicable at moderate power when the intraband carrier–carrier scattering times are small, typically less than 0.2 ps. This theory of laser-diode noise does not differ much from the three-level-atom theory, except for the fact that parameters that are unity for three-level-atom lasers acquire nonunity values.

The optical gain $\mathbf{G}$ is a nonlinear function of N. An often employed, plausible, expression for

G is $A(N - N_0)$, where N_0 is called the transparency carrier-number. An important parameter is the (dimensionless) differential gain

$$g \equiv \frac{N}{\mathbf{G}}\frac{d\mathbf{G}}{dN} = \frac{N}{N - N_0} \tag{69}$$

whose typical value is $g = 2$ (if $N = 2N_0$).

The population inversion factor n_p is no longer unity, a situation that would also occur for three-level atoms if there were some delay in the absorbing-state to ground-state decay. In semiconductors, it may be that neither the emitting nor the absorbing level is occupied by an electron or that both levels are occupied. In either situation, interaction with the optical field does not occur. Yet level pairs of the latter kind contribute to carrier dynamics. The well-known general expression for n_p is $\{1 - \exp[(h\nu - eU)/kT]\}^{-1}$. Note that the values of n_p are usually smaller than those derived from the expression $N/(N - N_0)$, which has been conjectured in early works on the basis of an analogy with two-level atoms. Short laser diodes operate at high carrier densities and n_p may not be much greater than unity even at room temperature. A typical value is $n_p = 1.5$.

Radiative spontaneous carrier recombination of three-level atoms is described by Equation 44. This expression applies to semiconductors only at low temperatures. At elevated temperatures the so-called 'bimolecular' approximation that assumes that the rate S is proportional to the square of N is plausible. At wavelengths of 1.3 μm and beyond, and room temperature, spontaneous carrier recombination is in fact dominated by the (phonon-assisted) Auger effect. Let us define in general the dimensionless spontaneous carrier recombination parameter

$$s \equiv \frac{N}{S}\frac{dS}{dN} \tag{70}$$

A typical value for long-wavelength lasers is $s = 2.5$.

Radiative spontaneous carrier recombination fluctuates at the shot-noise level. But this is not so for Auger recombination, contrary to widespread belief. We thus set

$$\mathbf{S}_s = \zeta S \tag{71}$$

with a typical value $\zeta = 2$.

The low-frequency RIN is obtained from rather straightforward modifications to the expressions in Section 5, or by setting $f = 0$ in equation 46 of [3]. The result for quiet pumps is

$$S\frac{\text{RIN}}{2} = \frac{1}{r^3}[(2 - r)(1 + r) + (\zeta - 1)r + 2(a - 1)] \tag{72a}$$

$$a \equiv n_p\left(\frac{s}{g}\right)^2 \tag{72b}$$

which reduces to Equation 66 when $\zeta = a = 1$, and $\xi = 0$. Just above threshold ($r \approx 0$), the RIN of a laser diode is a times the RIN of three-level-atom lasers. If we introduce in Equation 72 the typical parameter values quoted above (i.e. $\zeta = 2$, $a = 2.35$), the curve labelled 'semiconductor' in Fig. 5 is obtained.

For long scattering times (low temperatures), a hole forms in the spectral distribution, and the concept of electron or hole temperature no longer applies (spectral-hole burning, HB). Even when electrons and holes are in thermal equilibrium, their temperatures may be higher than that of the lattice (carrier heating, CH). Equilibrium between carriers and lattice is mainly ensured by longitudinal optical phonons. Because these phonons have an energy of the order

of 30 meV, their population is small at reduced temperatures and the time it takes for the carrier temperature to relax to the lattice temperature may exceed 10 ps. (See the textbooks [29].)

The net effect of HB and CH on laser operation is to introduce an explicit dependence of the optical gain on photon rate, called 'gain compression', essential to understanding the high-power density behaviour of laser diodes. Recent experiments [30] support the view that HB rather than CH is chiefly responsible for gain compression. But CH may be important for the phase-noise problem, not discussed in the present paper.

Other mechanisms (induced gratings, $1/f$ noise, spurious side-modes or TM modes, weak transverse guidance, scattering centres in the junction and weak external reflexions, carrier diffusion and capture times ...) have also been invoked to explain observed departures of laser-diode operation from standard-theory predictions. These latter effects are not discussed here.

Gain compression was first introduced by Channin [31] to explain the unexpectedly strong damping of relaxation oscillations, as an explicit dependence of the optical gain on photon number. Because little gain compression suffices to explain the observed damping, one may equivalently assume that the optical gain depends on photon *rate* rather than photon number. The difference between the two viewpoints becomes important at high power density, however, as previously discussed [4].

Statistical fluctuations of the optical gain should also be taken into account when gain compression is considered. They introduce a minimum level floor both to amplitude and phase fluctuations [5].

7. Phasor theory

The theory presented by Henry in [24] appears to coincide with the quantum result written in the normally-ordered form. We have shown in [3] and [4] that the expressions obtained from [24] for directly measurable quantities such as Δn and ΔQ coincide with that derived from the classical theory (then called 'circuit theory'). Agreement is proved in this section for the photon-number variance, a result not previously reported. The assumptions made in Section 4 are employed again here for the sake of simplicity.

Henry considers in place of the photon number m the 'intensity' **I** that has the same mean value ($\langle \mathbf{I} \rangle = \langle m \rangle$) but different fluctuations. The rate equations read

$$i\Omega\,\Delta n = -\frac{\Delta n}{\epsilon} - \Delta\mathbf{I} + F_n \tag{73a}$$

$$i\Omega\,\Delta\mathbf{I} = \frac{\Delta n}{\epsilon} + F_{\mathbf{I}} \tag{73b}$$

The (proper and cross) spectral densities of F_n and $F_{\mathbf{I}}$ are

$$\mathbf{S}_n = (1 + \xi)Q \qquad \mathbf{S}_{\mathbf{I}} = 2Q \qquad \mathbf{S}_{n\mathbf{I}} = -2Q \tag{73c}$$

Note that the cross-spectral density between F_n and $F_{\mathbf{I}}$ is twice as large as the one given earlier for F_n and F_m in Equation 49d.

Since the above rate equations are formally the same as those in Equation 49, the spectral density of $\Delta\mathbf{I}$ is given by Equation 51e with m changed to **I**:

$$F(\epsilon, \Omega^\circ)\mathbf{S}_{\Delta\mathbf{I}} = \mathbf{S}_n + (1 + \epsilon^2\Omega^{\circ 2})\mathbf{S}_{\mathbf{I}} + 2\mathbf{S}_{n\mathbf{I}} \tag{74}$$

When the spectral densities in Equation 73d are introduced into Equation 74, we obtain for the spectral density of $\Delta\mathbf{I}$

$$Q^{-1}\tau_{\mathrm{p}}^{-2}F(\epsilon,\Omega^{\circ})\mathbf{S}_{\Delta\mathrm{I}} = 1\xi + 2(1+\epsilon^2\Omega^{\circ 2}) - 4 = \xi - 1 + 2\epsilon^2\Omega^{\circ 2} \tag{75}$$

Note that the spectral density of $\mathbf{I}$ is negative at $\Omega = 0$ if $\xi < 1$. The calculations can nevertheless be pursued in a formal manner.

The variance of $\Delta\mathbf{I}$ obtained by integrating the spectral density given above over frequency is

$$\langle \Delta\mathbf{I}^2 \rangle = n + \tfrac{1}{2}m(\xi - 1) \tag{76a}$$

The variance of m is related to the intensity variance by [24]

$$\langle \Delta m^2 \rangle = \langle \Delta\mathbf{I}^2 \rangle + \langle \mathbf{I} \rangle = n + \tfrac{1}{2}m(\xi + 1) \tag{76b}$$

a result that coincides with Equation 50. The formalism is incomplete, however, since it does not indicate how to evaluate the spectral density of Δm (only its frequency integral is calculable).

The amplitude noise of the output light (i.e. of the detected current) is

$$\Delta Q = \Delta\mathbf{I} + u \qquad \mathbf{S}_u = Q \qquad \mathbf{S}_{\Delta\mathrm{I},u} = 0 \tag{77}$$

In other words, an independent fluctuation u, attributed to the 'detector shot-noise' is added to the 'light intensity' fluctuation expressed by the term $\Delta\mathbf{I}$. Accordingly, half the RIN is the spectral density of $\Delta\mathbf{I}/\mathbf{I}$, a fact that justifies the expression 'relative intensity noise'.

Note that if F_n were ignored we would find for QRIN/2 the value $+1$, while the correct result is -1. It is therefore not permissible to consider only the intensity equation Langevin term, as many authors assert. The error becomes even larger when the n_{p} factor is nonunity. It is only at high frequencies that F_n can be ignored.

The formalism in [24] clearly reproduces the quantum or classical results for the model considered. The question, however, is whether it can be justified on the basis of semiclassical 'phasor' arguments. The discussion below shows that this is not the case.

Let us first recall the (mathematically correct) argument leading to the expression $2Q$ of the spectral density of F_{I} in Equation 73c. The classical field is proportional to the square root of intensity $\mathbf{I}$. This intensity $\mathbf{I}$ is normalized in such a way that its average value is equal to the average photon number, as indicated above. Assume that at time t_k the field is incremented by a randomly phased field of modulus unity ('photon event'). The light intensity then becomes $|\sqrt{\mathbf{I}} + \exp(\mathrm{i}\theta_k)|^2$, where θ_k is a random variable evenly distributed between 0 and 2π. The increment of $\mathbf{I}$ at that time is therefore equal to $1 + 2\sqrt{\mathbf{I}}\cos(\theta_k)$. The '1' can be neglected here because $\mathbf{I}$ is a large number. The Langevin force relating to the intensity-evolution equation is thus a sum of δ-functions,

$$F_m = \sum_k 2\sqrt{\mathbf{I}}\cos(\theta_k)\,\delta(t - t_k) \tag{78}$$

Assuming that the 'photon events' occur at an average rate $1/\tau_{\mathrm{p}}$ (the statistics of this point-process turn out to be irrelevant) it is easily shown that the spectral density of F_{I} is indeed $2\mathbf{I}/\tau_{\mathrm{p}} = 2Q$.

The 'photon events', if real, would appear for large τ_{p} values as rare but intense bursts of the intracavity field. If the mean photon number is, say, 10 000, the increments are of the order of 100 photons. This, to our knowledge, has not been observed. Furthermore, the phasor picture defies one's physical intuition when the spectral density of the light-intensity fluctuations is negative.

In our formulation, the expression $2Q$ of F_m is understood instead as $2Q = Q + Q$, one Q expressing an exchange of photons with the emitter, while the other Q expresses an exchange of photons with the absorber. It is thus natural that the cross-Langevin force be equal to $-Q$, and not $-2Q$ as in the phasor picture. The expression $-2Q$ for $\mathbf{S}_{n\mathrm{I}}$ indeed rests on the concept that the photon-number bursts are tied to a corresponding decrease in electron number. But then one cannot understand why $\mathbf{S}_n = Q$, since, according to this picture, $\mathbf{S}_n$ should at least be equal to $2Q$. In fact the expression $\mathbf{S}_n = Q$ is obtained in [24] by an argument that is correct, but that cannot be employed consistently together with the phasor argument.

To conclude, the theory in [24] is accurate for the model considered, but it cannot be justified consistently on the basis of semiclassical arguments. The problem is that many authors applied the results in [24] to laser models for which it has not been justified, particularly for multiple and dispersive elements, gain guidance, gain compression, and so on. The classical theory of noise presented in this paper, because it is based on sound classical arguments, can be generalized safely.

8. Conclusion

We have presented a classical theory of laser noise that rests on a simple concept: transitions of atoms from the lower to the upper state (or the converse) are independent when they are subjected to a classically prescribed field. For treating amplitude noise it suffices to consider resonant atoms. But the same principle applied to detuned atoms leads to the noise sources required to treat phase noise. For the sake of brevity, only amplitude noise was considered. The expressions obtained are in exact agreement with the quantum results.

For pedagogical reasons, only simple laser models have been considered. Results for more complicated multielement lasers have been discussed elsewhere [32]. Many other problems (e.g. feedback [33] and laser start-up [34]) have been treated by alternative semiclassical or quantum methods. It is our intention to reconsider these problems in a unified manner.

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Appendix: Spectral densities

The spectral density of real, zero-mean, ergodic, stationary random functions is considered. For a complete discussion the reader may consult textbooks such as [35]. The quantities denoted here by x or y correspond to Δn, $\Delta Q, \ldots$ in the main text. Ergodicity implies that ensemble averages can be replaced by time averages. This is required since any practical measurement rests on time averaging. Stationarity means that the origin of time is irrelevant. We restrict ourselves to real functions of time because phase fluctuations are not discussed in the main text.

A random function is denoted $x(t, \zeta)$ where t denotes time and ζ belongs to a probability space. For each ζ-value, $x(t)$ represents a function of time, the ζ-argument being omitted for brevity. A random function is characterized by a probability $p(x_1, \ldots, x_n; t_1, \ldots, t_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$ with x between x_1 and $x_1 + \mathrm{d}x_1$ at time $t_1, \ldots$, and between x_n and $x_n + \mathrm{d}x_n$ at time t_n, where n is any integer. For two processes $x(t)$ and $y(t)$, a similar definition applies, the arguments of p now being x_k, y_k and t_k. If two random functions are independent (e.g. they have no causal connection) the probability factorizes into the product of a probability for x and a probability for y. The signs $\langle\ \rangle$ will indicate an ensemble average, i.e. an integral with respect to ζ with $p(\zeta)$ as a weighting factor.

The cross-covariance between $x(t)$ and $y(t)$ is

$$C_{xy}(\tau) \equiv \langle x(\tau)y(0)\rangle \tag{A1}$$

and $C_{xy}(\tau) = C_{yx}(-\tau)$ because of stationarity. Thus $C_x(\tau) \equiv C_{xx}(\tau)$ is an even function of τ

and the variance $C_x(0)$ is positive. Two random functions $x(t)$ and $y(t)$ are said to be uncorrelated if $C_{xy}(\tau) = 0$. This is the case if x and y are independent.

The cross-spectrum $\mathbf{S}_{xy}(f)$ is defined as the Fourier transform of the cross-covariance,

$$S_{xy}(f) \equiv \int_{-\infty}^{+\infty} C_{xy}(\tau)\exp(-2\pi i f\tau)\,\mathrm{d}\tau \tag{A2}$$

Since $C_{xy}(\tau)$ is real $\mathbf{S}_{xy}(0)$ is real. We also have that $\mathbf{S}_{xy}(f) = \mathbf{S}^*_{yx}(f)$, where the star denotes complex conjugation, and thus $\mathbf{S}_x(f) \equiv \mathbf{S}_{xx}(f)$ is real. Obviously the cross-spectrum of uncorrelated processes vanishes.

It follows from the linearity property of $C_{xy}(\tau)$ with respect to x and y that

$$\mathbf{S}_{ax+by,\,cx+dy} = ac^*\mathbf{S}_x + bd^*\mathbf{S}_y + ad^*\mathbf{S}_{xy} + bc^*\mathbf{S}^*_{xy} \tag{A3}$$

where the argument f is omitted. In particular, setting $a = c$ and $b = d$ we have

$$\mathbf{S}_{ax+by} = aa^*\mathbf{S}_x + bb^*\mathbf{S}_y + ab^*\mathbf{S}_{xy} + a^*b\mathbf{S}^*_{xy} \tag{A4}$$

Since $C_x(0)$ is positive, the integral of $\mathbf{S}_x$ over frequency from plus to minus infinity is positive.

If $x(t)$ is the input to a linear system with impulse response $h(t)$, the output is

$$y(t) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)\,\mathrm{d}\alpha \tag{A5}$$

Multiplying both sides by $x(t-\tau)$ and averaging we obtain

$$C_{yx}(\tau) = C_x(\tau) * h(\tau) \tag{A6}$$

where the star (middle position) denotes a convolution, or, taking the Fourier transform of Equation A6,

$$\mathbf{S}_{yx}(f) = \mathbf{S}_x(f)H(f) \tag{A7}$$

where $H(f)$ is the Fourier transform of $h(\tau)$, defined as in Equation A3.

If we now multiply both sides of Equation A5 by $y(t+\tau)$ and average we obtain analogously

$$\mathbf{S}_y(f) = \mathbf{S}_{yx}(f)H^*(f) \tag{A8}$$

Thus, from Equations A7 and A8,

$$\mathbf{S}_y(f) = \mathbf{S}_x(f)H(f)H^*(f) \tag{A9}$$

If $\mathbf{S}_x(f)$ were negative for some value of f we could find a narrowband filter of impulse response $h(t)$ such that $\mathbf{S}_y(f)$ would be negative over the full frequency range, and its integral would be negative, in contradiction with a previously established result. Thus the spectral density of any function of time must be nonnegative.