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Biorthogonality relations for bianisotropic media

J. A. Arnaud

Bell Telephone Laboratories, Inc., Crawford Hill Laboratory, Holmdel, New Jersey 07733

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Index Heading: Inhomogeneous media.

As is well known, the eikonal equation of geometrical optics is obtained by making the substitution $\nabla \to \nabla S$ in Maxwell's equation, where ∇ denotes the gradient operator and S the eikonal.[1] We have observed in a previous letter[2] that the wave equation obeyed by the scalar field ψ describing beam waves can be obtained by writing the eikonal equation in the form of a power series in $\mathbf{p} \equiv \partial S$, where ∂S denotes the gradient of S in a plane perpendicular to the system axis (z), ordering each term in symmetrical form, and making the reverse substitution $\nabla S \to \nabla$.

This procedure is quite similar to that used to obtain the Schroedinger equation from the Hamilton–Jacobi equation of classical mechanics.[3] In quantum mechanics, the hamiltonian operator is required to be hermitian, in order that the existence of the particle under consideration be preserved.[4] Because the optical field is described by a pair of vectors rather than by a scalar quantity, and because we are interested in lossy media that have no analog in mechanics, the optical problem is somewhat different from the mechanical problem and needs to be investigated. We will show that, in optics, the symmetrization procedure preserves in a natural way the biorthogonality relation existing between the vector fields. For generality, consideration is given here to bianisotropic media.[5]

If we assume that the medium is linear, time-invariant, free of time and space dispersion, and omit the $\exp(\kappa t)$ time dependence of the sources, Maxwell's equations, in a source-free region, are

$$\nabla \times \mathbf{e} = -\kappa \mathbf{b}, \tag{1a}$$

$$\nabla \times \mathbf{h} = \kappa \mathbf{d}. \tag{1b}$$

If we substitute in Eq. (1) field components of the form $e_0 \exp(S + \cdots)$, where e_0 is independent of κ, S is proportional to κ, and the omitted terms are of the order of $\kappa^{-1}, \kappa^{-2}, \ldots$, the leading terms in (1) are

$$\pi \times \mathbf{e}_0 = -\kappa \mathbf{b}_0, \tag{2a}$$

$$\pi \times \mathbf{h}_0 = \kappa \mathbf{d}_0, \tag{2b}$$

where $\pi \equiv \nabla S = (\kappa \mathbf{b}_0 \times \kappa \mathbf{d}_0)(\mathbf{e}_0 \cdot \kappa \mathbf{d}_0)^{-1}$. The complex ray vector $\mathbf{s} \equiv (\mathbf{h}_0 \times \mathbf{e}_0)(\mathbf{e}_0 \cdot \kappa \mathbf{d}_0)^{-1}$ satisfies the same equations as π, except for the substitutions $\kappa \mathbf{b}_0 \rightleftarrows \mathbf{h}_0$, $\kappa \mathbf{d}_0 \rightleftarrows \mathbf{e}_0$.

A convenient way of writing the linear relation between $\mathbf{e}$, $\mathbf{h}$, $\mathbf{d}$, and $\mathbf{b}$ is[6]

$$\begin{bmatrix} -\kappa \mathbf{d} \\ \mathbf{h} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \mathbf{e} \\ -\kappa \mathbf{b} \end{bmatrix}, \tag{3}$$

where $\mathbf{M}$ denotes a 6×6 matrix, a function of κ and of position. It is easy to show that Eqs. (2) and (3) have nontrivial solutions only if

$$\det \left\{ [\mathbf{1}\ \pi \times] \mathbf{M} \begin{bmatrix} \mathbf{1} \\ \pi \times \end{bmatrix} \right\} = 0, \tag{4}$$

where $\mathbf{1}$ stands for the 3×3 unit matrix and, for any vector $\mathbf{a}$, $(\pi \times)\mathbf{a} \equiv \pi \times \mathbf{a}$. Equation (4) is a partial differential equation for S, called the eikonal equation. From a previous remark, the complex ray vector $\mathbf{s}$ obeys the same Eq. (4) as π, $\mathbf{M}$ being changed to $\mathbf{M}^{-1}$. It is important to note that Eq. (4) remains the same if $\mathbf{M}$ is changed to its transposed $\tilde{\mathbf{M}}$, and π (or $\mathbf{s}$) to $-\pi$ (or $-\mathbf{s}$). Thus, to any complex ray trajectory in a medium characterized by a matrix $\mathbf{M}$, there corresponds an identical complex ray trajectory described in the opposite direction in the medium characterized by the matrix $\tilde{\mathbf{M}}$. Let us give an example of transposed media.

If a medium characterized by a matrix $\mathbf{M}$ in its rest frame is moving at a constant velocity $\mathbf{u}$, the material matrix seen by a fixed observer has the form[7]

$$\mathbf{M}(\mathbf{u}) = \tilde{\mathbf{T}}^{-1}(\mathbf{u}) \mathbf{M} \mathbf{T}(\mathbf{u}), \tag{5}$$

where $\mathbf{T}(\mathbf{u})\mathbf{T}(-\mathbf{u}) = \mathbf{1}$. Thus, if a medium is reciprocal in its own rest frame, i.e., if $\mathbf{M} = \tilde{\mathbf{M}}$, we have $\mathbf{M}(\mathbf{u}) = \tilde{\mathbf{M}}(-\mathbf{u})$. Two dielectric rods moving at the same velocity in opposite directions, for instance, are characterized by matrices that are the transposes of one another.[5]

We shall now let the z axis play a special role and rewrite Maxwell's Eqs. (1) in the transverse form, similar to the equations obtained by Marcuvitz and Schwinger[8] for the case of isotropic media. Considering also the equation formally adjoint to it, we have

$$\mathbf{H}(\boldsymbol{\partial},\mathbf{x},z)\boldsymbol{\psi}(\mathbf{x},z)+\partial_z\boldsymbol{\psi}(\mathbf{x},z)=0, \tag{6a}$$

$$\mathbf{H}^+(\boldsymbol{\partial},\mathbf{x},z)\boldsymbol{\psi}^+(\mathbf{x},z)-\partial_z\boldsymbol{\psi}^+(\mathbf{x},z)=0, \tag{6b}$$

$$\frac{d}{dz}(\boldsymbol{\psi}^+,\boldsymbol{\psi})\equiv\frac{d}{dz}\int\int_{-\infty}^{+\infty}\boldsymbol{\psi}^+\cdot\boldsymbol{\psi}\,dx_1\,dx_2=0, \tag{6c}$$

where $\mathbf{x}$ denotes a vector with components x_1, x_2; $\boldsymbol{\partial}$ has components $\partial/\partial x_1$, $\partial/\partial x_2$, and $\partial_z\equiv\partial/\partial z$. The explicit form of the 4×4 matrix operator $\mathbf{H}$ will not be given here. It can be obtained by writing Eq. (1) in the x_1, x_2, z cartesian-coordinate system, and rearranging the terms, using Eq. (3). In Eqs. (6), we have defined

$$\boldsymbol{\psi}\equiv\begin{bmatrix}\mathbf{v}\\ \mathbf{i}\end{bmatrix},\quad \mathbf{v}\equiv\begin{bmatrix}e_1\\ e_2\end{bmatrix},\quad \mathbf{i}\equiv\begin{bmatrix}h_2\\ -h_1\end{bmatrix}; \tag{7a}$$

$$\boldsymbol{\psi}^+\equiv\begin{bmatrix}-\mathbf{i}^+\\ \mathbf{v}^+\end{bmatrix},\quad \mathbf{v}^+\equiv\begin{bmatrix}e_1^+\\ e_2^+\end{bmatrix},\quad \mathbf{i}^+\equiv\begin{bmatrix}h_2^+\\ -h_1^+\end{bmatrix}. \tag{7b}$$

It can be shown that, physically, $\mathbf{e}^+$, $\mathbf{h}^+$ represent the fields of a wave propagating in the transposed medium, i.e., in the medium characterized by the matrix $\tilde{\mathbf{M}}$. The invariance relation Eq. (6c) is therefore a straightforward consequence of the Lorentz reciprocity theorem.[5,6]

We want now to relate the scalar wave function ψ used in beam optics to the vector wave function $\boldsymbol{\psi}$ that we have just defined in terms of the transverse components of the electric and magnetic fields $\mathbf{e}$, $\mathbf{h}$. For simplicity, we shall restrict ourselves to optical beams propagating in a direction close to the z axis. We view the medium as a perturbed stratified medium and assume that $\boldsymbol{\psi}$, $\boldsymbol{\psi}^+$ have the form

$$\boldsymbol{\psi}(\mathbf{x},z)=\boldsymbol{\Psi}(z)\psi(\mathbf{x},z), \tag{8a}$$

$$\boldsymbol{\psi}^+(\mathbf{x},z)=\boldsymbol{\Psi}^+(z)\psi^+(\mathbf{x},z), \tag{8b}$$

where the basal vector wave functions $\boldsymbol{\Psi}(z)$, $\boldsymbol{\Psi}^+(z)$ are of the form

$$\boldsymbol{\Psi}(z)=\boldsymbol{\Psi}_\alpha(z)\exp(\bar{\mathbf{p}}\cdot\mathbf{x})\exp\left[\int_0^z\gamma_\alpha(z)dz\right], \tag{9a}$$

$$\boldsymbol{\Psi}^+(z)=\boldsymbol{\Psi}_\alpha^+(z)\exp(-\bar{\mathbf{p}}\cdot\mathbf{x})\exp\left[-\int_0^z\gamma_\alpha(z)dz\right], \tag{9b}$$

and describe waves propagating in the stratified medium characterized by the matrices $\mathbf{M}(0,z)$ and $\tilde{\mathbf{M}}(0,z)$, respectively. $\boldsymbol{\Psi}_\alpha(z)$ and $\boldsymbol{\Psi}_\alpha^+(z)$ represent local eigenstates of the field and γ_α a local propagation constant. These quantities are defined by the eigenvalue equations

$$\mathbf{H}(\bar{\mathbf{p}},0,z)\boldsymbol{\Psi}_\alpha(z)+\gamma_\alpha(z)\boldsymbol{\Psi}_\alpha(z)=0, \tag{10a}$$

$$\mathbf{H}^+(-\bar{\mathbf{p}},0,z)\boldsymbol{\Psi}_\alpha^+(z)+\gamma_\alpha(z)\boldsymbol{\Psi}_\alpha^+(z)=0, \tag{10b}$$

where z plays the role of a parameter. Note that $\boldsymbol{\Psi}_\alpha$, $\boldsymbol{\Psi}_\alpha^+$ are defined only to within arbitrary scalar functions of z. Equation (6c), however, allows us to introduce the normalization conditions[9]

$$\boldsymbol{\Psi}_\alpha^+\cdot\boldsymbol{\Psi}_\beta=\begin{cases}1 & \text{if } \alpha=\beta,\\ 0 & \text{if } \alpha\neq\beta,\end{cases} \tag{10c}$$

$$\boldsymbol{\Psi}_\alpha^+\cdot d\boldsymbol{\Psi}_\alpha/dz=0,\quad \alpha,\beta=1,2,3,4. \tag{11}$$

$\boldsymbol{\Psi}_\alpha(z)$ and $\boldsymbol{\Psi}_\alpha^+(z)$ are now well-defined functions of z, except for a constant factor that is to be found from the initial conditions.

In writing Eqs. (9), we have assumed that the scale of variation of the medium parameters in the z direction is large compared with $(\gamma_\alpha-\gamma_\beta)^{-1}$, $\alpha\neq\beta$, and neglected the coupling between the four eigenstates of polarization. In what follows, we deal with only one eigenstate, namely, $\alpha=1$.

Let us now consider the scalar part of the vector wave function. From Eqs. (6c) and (8), we find that the invariance condition[10]

$$\frac{d}{dz}(\psi^+,\psi)=\frac{d}{dz}\int\int_{-\infty}^{+\infty}\psi^+(\mathbf{x},z)\psi(\mathbf{x},z)dx_1dx_2=0 \tag{12}$$

must hold for any ψ, ψ^+.

The eikonal equation applicable to ψ is obtained from the substitution $\boldsymbol{\nabla}\rightarrow\bar{\boldsymbol{\pi}}+\boldsymbol{\nabla}S$ (i.e., $\boldsymbol{\partial}\rightarrow\bar{\mathbf{p}}+\mathbf{p}$, $\partial_z\rightarrow\gamma_1+\partial_zS$) in Eq. (6a),

$$\det\{\mathbf{H}(\bar{\mathbf{p}}+\mathbf{p},\mathbf{x},z)+(\gamma_1+\partial_zS)\mathbf{1}\}=0, \tag{13}$$

where $\mathbf{1}$ denotes the 4×4 unit matrix. Equation (13) can be solved in principle for ∂_zS. The solution of Eq. (13) corresponding to the eigenstate of polarization selected before is rewritten

$$H(\mathbf{p},\mathbf{x},z)+\partial_zS=0. \tag{14}$$

Expanding H in power series of $\mathbf{p}$ and writing each term in symmetrical form, we get

$$\partial_zS+f(\mathbf{x})+[\mathbf{g}(\mathbf{x})\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{g}(\mathbf{x})]+\tfrac{1}{2}\tilde{\mathbf{p}}\mathbf{F}(\mathbf{x})\mathbf{p}+\cdots=0, \tag{15}$$

where f denotes a scalar, $\mathbf{g}$ a vector, and $\mathbf{F}$ a symmetrical matrix. They are functions of $\mathbf{x}$ and z (the z dependence is omitted for brevity). Replacing now $\boldsymbol{\nabla}S$ in Eq. (15) by the operator $\boldsymbol{\nabla}$, we obtain

$$\partial_z\psi+\{f(\mathbf{x})+[\mathbf{g}(\mathbf{x})\cdot\boldsymbol{\partial}+\boldsymbol{\partial}\cdot\mathbf{g}(\mathbf{x})]+\tfrac{1}{2}\tilde{\boldsymbol{\partial}}\mathbf{F}(\mathbf{x})\boldsymbol{\partial}+\cdots\}\psi=0, \tag{16a}$$

and, similarly, for the adjoint equation

$$\partial_z\psi^+-\{f(\mathbf{x})-[\mathbf{g}(\mathbf{x})\cdot\boldsymbol{\partial}+\boldsymbol{\partial}\cdot\mathbf{g}(\mathbf{x})]+\tfrac{1}{2}\tilde{\boldsymbol{\partial}}\mathbf{F}(\mathbf{x})\boldsymbol{\partial}+\cdots\}\psi^+=0. \tag{16b}$$

It is not difficult to show that a solution ψ of Eq. (16a) and a solution ψ^+ of Eq. (16b) satisfy the invariance relation Eq. (12). This follows from the fact that $\boldsymbol{\partial}$ is an

antisymmetrical operator. We have, for instance, for the component $\partial_1\equiv\partial/\partial x_1$ of $\boldsymbol{\partial}$,

$$(\psi^+,\partial_1\psi)+(\partial_1\psi^+,\psi)$$

$$\equiv\int\int_{-\infty}^{+\infty}(\psi^+\partial\psi/\partial x_1+\psi\partial\psi^+/\partial x_1)dx_1dx_2$$

$$=\int_{-\infty}^{+\infty}dx_2\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_1}(\psi^+\psi)dx_1=0, \tag{17}$$

assuming that the product $\psi^+\psi$ vanishes sufficiently rapidly at infinity.

In conclusion, we see that the vector field $(\mathbf{e},\mathbf{h})$ can be related in a well-defined way to the scalar wave function ψ used in beam optics, in the general case of bianisotropic media. The vector wave functions defined in Eq. (8) are approximate solutions of Maxwell's equations that are valid only when the wave remains confined to the neighborhood of the z axis, and when the coupling between the local eigenstates of polarization can be neglected. We have also observed that symmetrization of the eikonal equation ensures the applicability of the Lorentz reciprocity theorem to the solutions of the wave equation that is derived from it.

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[6] J. A. Arnaud and A. A. M. Saleh, Proc. Inst. Electr. Eng. **60**, 639 (1972). Note that the matrix **M** differs by a factor i from the one considered in the present paper.

[7] J. A. Arnaud and A. A. M. Saleh (unpublished). It is further shown in this work that, because $\mathbf{T}(\mathbf{u})\mathbf{T}^*(\mathbf{u})=\mathbf{1}$, the power state of a medium (that is, its losslessness, passivity, activity, indefiniteness) does not depend on **u**.

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[9] See, for instance, R. W. Hougardy and D. S. Saxon, in *Electromagnetic Theory and Antennas*, edited by E. C. Jordan (Macmillan, New York, 1963), p. 635.

[10] J. A. Arnaud, in *Progress in Optics XI*, edited by E. Wolf (North-Holland, Amsterdam, 1973).