Accuracy of Petermann's K-Factor in the Theory of Semiconductor Lasers

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Abstract—Petermann has proposed that the classical formula for the linewidth of a laser be multiplied by a factor $K \gg 1$ in the case of gain-guided semiconductor lasers [1]. The concept of power in the mode used by that author, however, is not well defined in a waveguide with gain [2], and his theory is therefore opened to question [3]. The analysis given here avoids this difficulty and nevertheless agrees with Petermann's result. This is because spatial mode filtering is strong in oscillating lasers.

I. Introduction

PETERMANN has proposed that the classical Schawlow–Townes expression for the linewidth of a laser be multiplied by a factor $K \gg 1$ in gain-guided semiconductor lasers [1]. This is because the fundamental mode of propagation has a curved diverging wavefront in the junction plane, and spontaneous emission from the active region couples more strongly to such a mode than to modes with plane wavefronts.

The concept of "power in a mode" for a multimoded waveguide with gain (or loss), however, is ambiguous [2]. Indeed, if we define the power of a mode as the integral of the modulus square of the modal field over the waveguide cross section, the total power is not the sum of the powers of the various modes. In other words, the modes are not power-orthogonal. This is so even if the waveguide supports nominally only one mode because radiation modes always exist in open structures. The validity of Petermann's calculations can therefore be questioned. In fact, an alternative definition of "power in the mode" in [3] virtually amounts to suppressing the K-factor. On the experimental side, the validity of Petermann's result is supported by many observations concerning gain-guided lasers. In the present letter, we will consider only the spectral width of nominally single-mode lasers.

II. The Laser Model

We shall discuss the idealized laser model represented in Fig. 1(a). The active slab has length L in the z-direction and width w in the y-direction. Its thickness $2d$ is assumed to be so small that the field is almost a constant within the slab. Let us set

$$i\,(k^2 - k_c^2)\,d \equiv u_0 \equiv b_o + ia_o \tag{1}$$

where k is the complex wavenumber of the active slab and k_c the wavenumber of the outer medium

$$k_c = k_0 + i\alpha \tag{2}$$

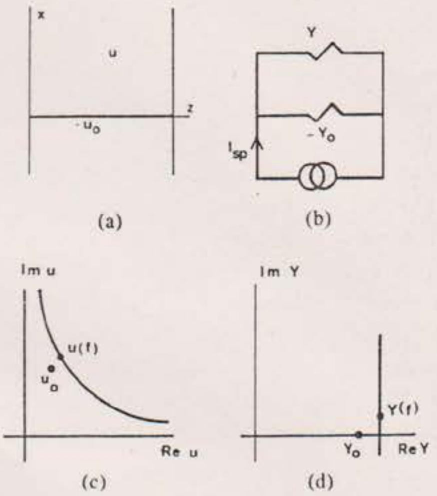

Fig. 1. (a) Schematic representation of the semiconductor laser model, with the active thin slab of negative admittance u_0, length L, and width w, in a lossy medium, terminated by perfectly reflecting end mirrors. (b) The classical representation of a single-mode oscillator. I_{sp} models the spontaneous emission. (c) Variation of u with frequency in the complex plane. (d) Variation of Y with frequency in the complex plane.

where α expresses a loss. This loss may model the coupling loss to some external detector of radiation spread out uniformly in the outer medium, an approximation often made in semiconductor laser theory. In (1), $a_o > 0$ expresses the fact that the slab provides normal (but weak) real-index guidance, while $b_o > 0$ expresses the stimulated emission gain in the slab material.

We shall consider in detail only TE waves with the electric field vector $\bar{E}$ directed along the y-axis and independent of y for $0 < y < w$. This field component $E_y = E(x, z)$ obeys the scalar Helmholtz equation, with a source term originating from spontaneous emission to be discussed later.

Because the end mirrors are assumed perfectly reflecting, the field vanishes at $z = 0$ and $z = L$. Thus, to within an amplitude factor,

$$E(x, z) = \exp(iu_l|x|)\sin(\beta_l z) \tag{3}$$

where

$$u_l^2 + \beta_l^2 = k_c^2;\ \beta_l = \pi l/L. \tag{4}$$

The integer L is the longitudinal mode number. For definiteness, we set

$$u = b + ia; \qquad b > 0 \tag{5}$$

where the l-subscripts have been dropped for brevity. At $x = 0$, E is a real function of z because obviously β_l is a

Manuscript received June 4, 1985; revised November 20, 1985.
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IEEE Log Number 8407241.

0018-9197/86/0400-0505$01.00 © 1986 IEEE

real quantity. Therefore, the longitudinal modes presently considered, corresponding to various l-values, are power-orthogonal, while the transverse modes usually considered in laser theory are not power-orthogonal.

Before evaluating the laser linewidth, let us recall Yariv's derivation of the linewidth of a single-mode laser [4].

III. Classical Derivation of Single-Mode Laser Linewidth

To facilitate a subsequent comparison to the multimode case, minor changes of notation from [4] are made. The laser is modeled by the electrical circuit shown in Fig. 1 (b), which consists of a passive admittance $Y = G + iB$ and an active admittance $-Y_0 = -(G_0 + iB_0)$. In normal operation Y is very close to Y_0. G and G_0 are both positive quantities, $G > 0$ expressing the loss (perhaps the coupling loss), while $G_0 > 0$ expresses the stimulated emission gain. This stimulated emission is accompanied by a spontaneous emission modeled by an electrical current

$$I_{sp} = (4hfG_0\,df)^{1/2} \tag{6}$$

for the optical frequency range f to $f + df$. We have assumed complete population inversion.

Using classical circuit theory it is straightforward to show that the spectral power density in the load Y is

$$S(f) = 4hfG_0G/|Y - Y_0|^2. \tag{7}$$

$Y_0 = G_0$ is almost independent of frequency, while Y consists of a constant real part G and an imaginary part B which varies rapidly with frequency because of the cavity resonance, as Fig. 1 (d) shows.

The critical term in (7) is the denominator $|Y - Y_0|^2$ which leads to the usual Lorentzian line shape, because of the linear variation of B with f in the neighborhood of the cavity resonance. The total power P_c in the load is essentially the product of the peak S-value (for $B = 0$) and the laser linewidth Δf. It follows that the product $P_c\Delta f$ is a constant

$$P_c\Delta f = 2\pi hf(\Delta f_c)^2 \tag{8}$$

where Δf_c is the full width of the passive ("cold") cavity resonance. Equation (8) is, to within a factor of two due to saturation effects, the classical Schalow–Townes formula.

Let us now see what goes on differently in the case of our weakly-guiding laser model.

IV. Spectral Width of Weakly Guiding Lasers

Interestingly enough we can use the simple circuit model discussed above for each longitudinal mode (l-mode). It is well known that propagation in a stratified medium is analogous to a transmission line problem. According to [5], $u/2\pi f\mu_0$, where μ_0 represents the free-space permeability, can be viewed as the characteristic admittance of the transmission line representing the outer medium. Because this medium extends to infinity and is lossy, this is also the load admittance. Likewise, $u_0/2\pi f\mu_0$ is the negative of the active admittance representing the thin active slab located at $x = 0$. The current I_{sp} modeling spontaneous emission is related to the real part of $u_0/2\pi f\mu_0$ exactly as in the classical case treated earlier.

The spectral density in our laser model is, therefore, replacing Y by u and Y_0 by u_0 in (7)

$$S'(f) = 4hfb_ob/|u - u_0|^2. \tag{9}$$

This result can also be obtained by modeling spontaneous emission by δ-correlated currents in the y-z plane and performing Fourier series transforms.

The only difference between (9) and (7) comes from the way the quantity $|u - u_0|$ varies with frequency. This is illustrated in Fig. (c) and (d). These figures show the variations of the real and imaginary parts of u or Y as the frequency varies. In that complex plane, the quantity $|u - u_0|$ is the geometrical distance between the point u and the point u_0. In our laser model, the relation

$$u^2 + \beta_l^2 = (k_0 + i\alpha)^2; \qquad u \equiv b + ia \tag{10}$$

implies that in the complex u-plane

$$ba = \alpha k_0 \tag{11}$$

since β_l is real. Equation (11) shows that the u-curve in Fig. 1 (c) is an hyperbola. Only the slope of that hyperbola near u_0 matters when $u \approx u_0$, a relation applicable when the laser is well above threshold. In (11), α and k_0 can be viewed as constants.

It is a simple geometrical matter to show that, when $u \approx u_0$, the classical result in (8) is multiplied by the factor

$$1 + (b_o/a_o)^2. \tag{12}$$

This linewidth enhancement factor can now be identified as Petermann K-factor. Indeed, the x-variation of the field of the fundamental transverse mode in (3) is approximately: $\exp(iu_0|x|)$, and therefore [7]

$$K = \left| \int_{-\infty}^{+\infty} |E^2(x)|\,dx \Big/ \int_{-\infty}^{+\infty} E^2(x)\,dx \right|^2 = 1 + (b_o/a_o)^2. \tag{13}$$

Thus, in the limit of large injected currents, the laser linewidth is increased by the K-factor, as asserted in [1] and [8]. This K-factor is much larger than unity in the junction plane for gain-guided lasers, and in the plane perpendicular to the junction if the guidance is weak as presently considered.

In the previous discussion, only TE waves were considered. But spontaneous emission is modeled by three uncorrelated xyz components. The J_y component excites the TE waves just discussed while the J_x and J_z components both excite TM waves. When these other two components are taken into account the complete expression for the spectral density is found

$$S_{nl}(f) = 4\,hf(b_ob/k_0^2)\,\{k_0^2|u - u_0|^{-2} + \beta^2|u - u_0|^{-2} + 1\} \tag{14}$$

where the subscripts n (transverse y-mode number) and l (longitudinal z-mode number) have been omitted on u and β for brevity. We will not calculate here the two additional terms in (14) but only point out that when the active slab has negligible guidance: $u_0 \approx 0$, (14) gives isotropic radiation as it should (LED operation).

V. Laser Linewidth Near Threshold

We have given a proof that well above threshold the classical formula for the laser linewidth should be multiplied by the K-factor. This proof is not subjected to the same objections as the calculation in [1]. It remains to answer the question: what is the accuracy of the theory near threshold? To answer that question, we have solved [14] numerically for various injected currents.

To do that, we first select some $b_o(f)$ curve, a_o being assumed independent of frequency. Equation (14) gives the spectral density of a mode of order n, l. We select the strongest mode (called the "oscillating" mode), corresponding to $n = 0$ and an l-value of the order of k_0L/π, where k_0 relates to the peak frequency of the $b_o(f)$ curve. We then evaluate the width Δf of that $S(f)$ curve. This is the laser linewidth. Next we evaluate the power P_c in the oscillating mode by integration over frequency. A numerical integration is required because the profile is not exactly Lorentzian. Then we define an effective linewidth enhancement factor K' as the product $\Delta f\,P_c$ divided by the classical expression in (8).

In order to calculate the injected current corresponding to the $b_o(f)$ curve initially selected, we evaluate the power in all the n, l modes, say P. Because in our idealized laser model there is no loss of electrons, holes, or photons, the injected current is simply equal to the total optical power P, to within a constant factor

$$I/e = P/hf. \tag{17}$$

Note that in this expression the variation of f can be neglected. For a complete laser design, the initial $b_o(f)$ curve should be selected according to the Fermi–Dirac distribution, which depends on temperature and on the carrier density. However, the shape of the gain curve does not vary much in practice as we go through threshold, and it is therefore permissible to select a fixed shape (an inverted parabola in the numerical examples to be given later), changing only the magnitude of the peak value.

To appreciate the significance of the currents obtained, it is useful to evaluate two particularly significant injected currents, namely the threshold current I_{th}, and the single longitudinal-mode current I_l to be defined later. The threshold current is defined by the condition that the gain equals the loss. The classical expression [4, p. 182] can be written

$$I_{th} = 8\pi\,e\delta f(S/\lambda^2)\,\alpha L \tag{18}$$

where e is the electronic charge, δf the laser gain linewidth (on the order of $0.1f$), S the mode area in the xy plane, λ the wavelength in the medium, and αL the single pass loss (or gain). Numerical results are given in the next section.

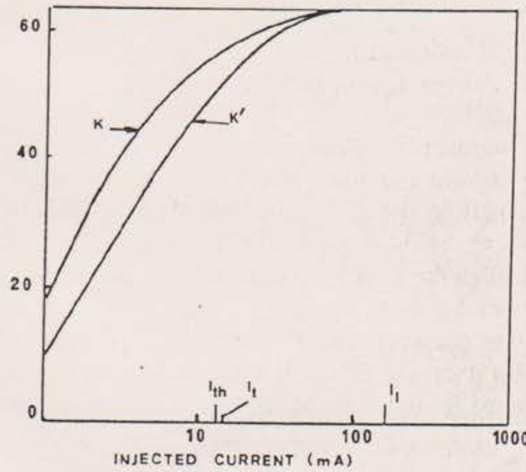

Fig. 2. Variation of the effective linewidth enhancement factor K' as a function of injected current I. For large I/I_{th} ratio, K' is almost equal to the K-factor, also shown. On the horizontal axis are marked the threshold current I_{th} and the single longitudinal-mode current I_l, defined in the text. Near threshold there is some discrepancy between K' and K because of imperfect transverse mode filtering. At I_t, half the power is in one transverse mode.

The "single longitudinal mode current," I_l, is defined by the condition that half the total optical power is in a single n, l mode. For that current, the coupling efficiency to a single-mode fiber is at most 50 percent. We are here referring to a transverse (prism-like) coupling in which only one longitudinal mode can be coupled, and not to the usual end mirror coupling. This I_l current will be found to be very high compared to the threshold current because gain-guided lasers provide very poor longitudinal mode discrimination, unless of course some distributed feedback mechanism is implemented.

VI. Numerical Results

We have selected the following values for the parameters that enter into (14)

$$L = 200\ \mu\text{m} \tag{19a}$$

$$w = 5\ \mu\text{m} \tag{19b}$$

$$\alpha = 0.005\ \mu\text{m}^{-1} \tag{19c}$$

$$a_o = 0.01 \times 2\pi\ \mu\text{m}^{-1} \tag{19d}$$

$$b_o = b_{om}\,\{1 - [10(k_0/2\pi - 1)]^2\}. \tag{19e}$$

The last expression shows that the $b_o(f)$ curve peaks at a medium wavelength of 1 μm. The width δf of that curve (defined by the condition that δf times the peak gain equals the gain curve area) is 40 THz. The mode area $w/2a_o = 80\ \mu\text{m}^2$, and the single pass loss $\alpha L = 1$. From these numerical values we calculate a threshold current of 13 mA, using (18).

Using the procedure described in the previous section varying the b_{om} parameter in (19e), we calculate the linewidth enhancement factor K' and the corresponding injected current I. K' is plotted in Fig. 2 as a function of I.

On the horizontal axis we have also shown the "single longitudinal-mode current," I_l, defined earlier, and the threshold current I_{th}. At I_l, half the power is in one transverse mode.

These numerical calculations first verify that at large injected current the linewidth enhancement factor K' is almost equal to the K factor, whose value here is $1 + (b_{om}/a_o)^2 = 64.3$. Indeed, the maximum b_{om} value is reached when the gain $a_o b_o/k_0$ equals the loss α. This condition gives $b_{om} = 0.5\ \mu m^{-1}$.

At lower currents, the K-factor decreases slightly because of a decrease of b_o. But the important point is the departure of K' from K. At threshold, for example, K' is significantly smaller than K.

VII. Conclusion

Using a somewhat academic, but well-defined model, we have shown that Petermann theory [1] predicts correctly the laser linewidth well above threshold in spite of formal objections. A correction needs to be introduced, however, near threshold. In the present letter we have left aside Henry α-factor. We have shown in [8] how this factor (or rather a modified version of it) combines with the K-factor. Clearly, our model can be generalized to account for a reflection at some distance from the active slab. The role of K in such a configuration will be discussed elsewhere. Another interesting problem, not treated here, is the role of K on the injection locking bandwidth and on the residual slave-laser linewidth.

Acknowledgment

J. Arnaud wishes to express his thanks to K. Petermann for a useful discussion.

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