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A theory of Gaussian pulse propagation

J. ARNAUD
Équipe de Microoptoélectronique de Montpellier, USTL, Pl. E. Bataillon, 34 060 Montpellier Cédex, France

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The complex-ray representation of Gaussian beams proposed by the author in 1968 is applied to the propagation of pulses with Gaussian envelope. Linear propagation in uniform time-invariant waveguides is first considered. Next, closed-form soliton solutions are obtained for a special kind of nonlinearity.

1. Introduction

It was shown by this author in 1968 [1–3] that the propagation of Gaussian beams is most easily treated by using a complex-ray representation*. This representation was soon after extended to anisotropic media [4]. It was subsequently pointed out that the propagation of Gaussian pulses could be treated along the same lines because of the formal analogy that exists between space and time ($x \to t$, $k_x \to -\omega$) [5]. The details, however, were not given. I now wish to show in detail how one can use the complex ray representation to treat the spreading of Gaussian pulses in dispersive media. In the last sections, we consider a medium with a special kind of nonlinearity, namely a logarithmic nonlinearity. We then show that stationary Gaussian pulses can be found, provided $d^2k/d\omega^2 < 0$. This paper treats comparatively simple problems for the sake of clarity, but the formalism used here is powerful enough to handle considerably more complicated media, for example anisotropic media.

2. Space – time ray optics

Let us first consider a time-harmonic plane wave at an angular frequency ω propagating along the z-axis. We denote the field

$$\begin{aligned} \psi(z, t) &= \psi_0 \cos(\varphi + kz - \omega t) \\ &= \text{Real Part } \psi \exp[i(kz - \omega t)] \\ \psi &= \psi_0 \exp(i\varphi) \end{aligned} \tag{1}$$

The propagation constant k is considered as a function of ω, which we approximate by a parabolic function

$$k = k_0 + a(\omega - \omega_0) + \tfrac{1}{2}b(\omega - \omega_0)^2 \tag{2}$$

about the angular frequency ω_0. The parameter $a = v_g^{-1}$ where $v_g = d\omega/dk$ and $\omega = \omega_0$ denotes the group velocity at ω_0 and $b = d^2k/d\omega^2$ where $\omega = \omega_0$ expresses the dispersion of the medium, i.e. the fact that the group velocity depends on ω.

At some frequency ω different from ω_0, the world-lines (or space–time rays) have a slope reciprocal of the group velocity i.e.

$$t/z = dk/d\omega = a + b(\omega - \omega_0) \tag{3}$$

Consider next a pulse whose duration involves a fairly large number of oscillations of the field. The frequency spectrum is no longer infinitely narrow and the various frequency components travel at

*This representation was presented first in the footnote on p. 190 of Reference 2 submitted in May 1968. There are apparently no earlier works published on the subject in Western literature. See also the footnote on p. 191 of Reference 3 which states the principle of a complex coordinate shift.

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different speeds, as Equation 3 shows. There is one particular frequency that reaches a specified point in space–time (t, z). It is given by Equation 3. However, the intensity is significant only if t/z is not too different from $a = v_g^{-1}$, because the frequency spectrum of the pulse is narrow.

We are mostly interested in the phaseshift $S(z, t)$ from the origin $(t = z = 0)$ to the point (t, z). Obviously, from Equation 1

$$S(z, t) = k(\omega)z - \omega t \tag{4}$$

If we replace $k(\omega)$ by its expression from Equation 2 and take ω from Equation 3,we obtain after rearranging

$$S = k_0 z - \omega_0 t - \tfrac{1}{2}(t - az)^2/bz \tag{5a}$$

To within a comparatively unimportant amplitude factor, the field that reaches the space–time point (t, z) is

$$\psi(z, t) = \exp\,[\mathrm{i}S(z, t)] \tag{5b}$$

We are now in position to deal with Gaussian pulse propagation in linear dispersive media.

3. Gaussian pulses

Because the medium considered is homogeneous (z-invariant) and time-invariant, the expression for $\psi(z, t)$ obtained above remains a solution of the wave equation if we add to z and t arbitrary constants. This indeed merely amounts to translating the origin of the z and t axes. New physics is obtained, however, if the constants added are complex (see Reference 3, footnote on p. 1910). $\psi(z, t)$ remains a solution of the wave equation, but the transformation cannot any longer be interpreted as a translation.

In order to get simple initial conditions, we perform imaginary shifts along both z and t in the direction of average motion, that is

$$z \to z + \mathrm{i}\sigma_0^2/b \tag{6}$$

$$t \to t + \mathrm{i}a\sigma_0^2/b \tag{7}$$

where σ_0 is as yet unspecified. The field expression becomes, dropping the time-independent terms from Equation 5b

$$\psi(z, t) = \exp\,[\mathrm{i}(k_0 z - \omega_0 t)]\,\exp\left[-\frac{1}{2}\,\mathrm{i}\,\frac{(t - az)^2}{bz + \mathrm{i}\sigma_0^2}\right] \tag{8}$$

We are mostly interested in the pulse envelope amplitude given by the real part of the argument of the second exponential term in Equation 8. This is

$$|\psi(z, t)|^2 = \exp\,[-(t - az)^2/\sigma^2] \tag{9}$$

where

$$\sigma^2 = \sigma_0^2 + b^2z^2/\sigma_0^2 \tag{10}$$

Clearly, Equations 9 and 10 represent a pulse with a Gaussian envelope of width σ_0 at $z = 0$ and σ at any z. This pulse is centred at $t = z/v_g$ as one expects, that is, the pulse travels at the group velocity applicable to the carrier frequency ω_0. The result in Equations 9 and 10 is of course well known. But it is usually obtained through a double Fourier transform, while the derivation given here is simple. Note that it was essential, in order to obtain Equation 9, to perform the imaginary translation along the line of average motion $t = z/v_g$.

Inside the Gaussian envelope the pulse is chirped, that is, the frequency varies. The pulse chirping is given by the imaginary part of the argument of the second exponential term in Equation 8. The instantaneous frequency

$$\omega(t) = -\frac{\partial\phi}{\partial t} = \omega_0 + \frac{bz}{b^2z^2 + \sigma_0^4}(t - az) \tag{11}$$

exhibits a linear variation with time within the pulse.

There are obvious analogies between the spreading of Gaussian pulses in time and the diffraction of Gaussian beams in free space. It must not be forgotten, however, that the spatial analog of pulse spreading is beam propagation in anisotropic media, while free-space is isotropic.

A further comment is in order: instead of expanding k up to second order in $\omega - \omega_0$ as was done in Equation 2, it is possible to expand ω up to second order in $k - k_0$. It can be shown that the final result is the same provided z be replaced by $v_g t$ in the denominator of the quadratic terms. This is permissible because the pulse energy is significant only in the neighbourhood of $z = v_g t$. Thus, the lack of symmetry between space and time in our result is only apparent.

4. General formulation

It is instructive to show that the results of the previous section can be recovered from the mode-generating concept, because this concept is applicable to inhomogeneous and time-varying media, while the imaginary translation used in Section 2 is much more restricted. The mode-generating concept is explained in Section 2–16 of Reference 4. In the present situation we need the linear terms exhibited in Equation 2.208 of that book. On the other hand, we can set $\alpha = 0$ because we are interested only in the fundamental mode. The formal space – time analogy would imply that we set $k_x = p \to -\omega$ in the equations. However, the algebra is somewhat simplified if we set instead $p = \omega - \omega_0$. Then, if S_g denotes the generating phase we have from the definition of ω: $p = -\partial S_g/\partial t$. With that notation, Equation 2 is

$$k = k_0 + ap + \tfrac{1}{2}bp^2 \tag{12}$$

The generating phase S_g is taken to be of the form

$$S_g = d_g + u_g t + \tfrac{1}{2}U_g t^2 \tag{13}$$

where d_g, u_g and U_g are unknown functions of z. From the definition of k $(= k_z)$ we have

$$\frac{\partial S_g}{\partial z} = k \tag{14}$$

If upper dots denote derivatives with respect to z, Equation 14 becomes, using Equations 12 and 13

$$\dot{d}_g + \dot{u}_g t + \tfrac{1}{2}\dot{U}_g t^2 = k_0 + ap + \tfrac{1}{2}bp^2 \tag{15}$$

where, from Equation 13

$$p = -\frac{\partial S_g}{\partial t} = -u_g - U_g t \tag{16}$$

If we substitute this expression for p into Equation 15 we obtain

$$\dot{d}_g + \dot{u}_g t + \tfrac{1}{2}\dot{U}_g t^2 = k_0 - a(u_g + U_g t) + \tfrac{1}{2}b(u_g + U_g t)^2 \tag{17}$$

It remains to identify both sides of Equation 15. Let us first consider terms proportional to t^2. We have

$$\dot{U}_g = bU_g^2 \tag{18}$$

The solution of this differential equation is straightforward

$$U_g = -1/(bz + \mathrm{i}\sigma_0^2) \tag{19}$$

where we have written the arbitrary constant of integration $\mathrm{i}\sigma_0^2$ to satisfy the boundary conditions at $z = 0$.

Next, consider the terms proportional to t in Equation 17. u_g obeys the differential equation

$$\dot{u}_g = -aU_g + bu_gU_g \tag{20}$$

Let us substitute the result in Equation 19 into Equation 20 and integrate. We obtain

$$u_g = az/(bz + \mathrm{i}\sigma_0^2) \tag{21}$$

if we select again the integration constant to satisfy the boundary condition at $z = 0$. The result in Equation 21 can be verified by direct substitution into Equation 20. Thus, leaving aside t-independent terms and the term $\exp(-\mathrm{i}\omega_0 t)$, we finally obtain the field

$$\psi(z,t) = \exp[\mathrm{i}(u_g t + \tfrac{1}{2}U_g t^2)] \tag{22}$$

$$= \exp[-\tfrac{1}{2}\mathrm{i}(t-az)^2/(bz+\mathrm{i}\sigma_0^2)] \tag{23}$$

which is identical to the Gaussian pulse solution in Equation 8.

As indicated earlier, this second, apparently more complicated approach, is needed to treat inhomogenous time-varying media such as the one encountered in the final section of this paper.

5. Nonlinear logarithmic media

At high optical intensities, the refractive index of a medium often increases through, for example, the optical Kerr effect. One usually postulates a linear dependence of the refractive index n on the intensity $I = |\psi|^2$

$$n(I) = n_0 + n_2 I \tag{24a}$$

Here we shall postulate instead a logarithmic law

$$n(I) = n_0 + n_2 \bar{I} \log(I/\bar{I}) \tag{24b}$$

The logarithmic law expresses well a saturation of the nonlinear effect at high intensities. The constants in Equation 24b are selected so that n and $\mathrm{d}n/\mathrm{d}I$ from the two laws in Equations 24a and b agree with each other at some given intensity $\bar{I} \ll 1/n_2$.

Since the logarithmic law diverges as $I \to 0$, some truncation is necessary, but it takes place only at extremely low values of the optical intensity, because of the condition: $\bar{I} \ll 1/n_2$. Thus, it is believed that the law in Equation 24, and the results to be derived are realistic for some media. The logarithmic nonlinearity is appealing from a theoretical standpoint because, in such a medium, a Gaussian beam or Gaussian pulse, gives rise to a square-law medium, that is a medium where the refractive index is at most a quadratic function of the space–time coordinates. Conversely, a square-law medium admits Gaussian field solutions, hence the possibility of finding closed form soliton solutions.

Let us look for an invariant solution of intensity

$$I = |\psi(z,t)|^2 = I_1 \exp[-(t-az)^2/\sigma_0^2] \tag{25}$$

where I_1 and σ_0 are constants to be determined. For such a solution, the refractive index is given by Equation 26 by inserting Equation 25 into Equation 24b with $I_1 \approx \bar{I} \ll 1/n_2$

$$n(z,t) = n_0 - n_2 \bar{I}(t-az)^2/\sigma_0^2 \tag{26}$$

Thus the local wavenumber k_0 in Equation 2 should be replaced by a space and time-varying quantity

$$k_0(z,t) = (\omega_0/c)n(z,t) = k_0 - \tfrac{1}{2}k_0\Omega^2(t-az)^2 \tag{27}$$

where k_0 now denotes a constant

$$k_0 = (\omega_0/c)n_0 \tag{28a}$$

and

$$\Omega^2 = 2(n_2/n_0)\bar{I}/\sigma_0^2 \tag{28b}$$

It remains to see whether a solution of the form in Equations 25a and b can be found in a medium described by Equation 27 for some values of I_1 and σ_0. The analysis is similar to the search for stationary solutions in focusing media (see, for example, the section 'uniform fibres' of Section 2.16 of Reference 4).

We shall proceed as in the previous section and only consider quadratic terms. Equation 18 now involves the focusing strength. $U_g(z)$ must satisfy the differential equation

$$\dot{U}_g + k_0\Omega^2 - bU_g^2 = 0 \tag{29}$$

The stationary solution sought for is obtained by setting $\dot{U}_g = 0$ in Equation 29. Thus

$$U_g = \mathrm{i}(-k_0/b)^{1/2}\Omega \tag{30}$$

The minus sign has been introduced inside the square root in Equation 30, anticipating that b must be negative.

The field intensity is, from Equations 30 and 13

$$I = |\exp(\mathrm{i}S_g)|^2 = \exp[-(-k_0/b)^{1/2}\Omega(t-az)^2] \tag{31}$$

to within a constant. This expression agrees with the solution we started from, Equation 25, provided

$$1/\sigma_0^2 = (-k_0/b)^{1/2}\Omega \tag{32}$$

Equation 32 is in fact identical to Equation 2.189 of Reference 4 with the replacement $\bar{\xi}_0 \to \sigma_0$ and $\gamma \to -b$, due to a change of notation in going from space to time variables.

But we must remember now that Ω and k_0 are related to the postulated values of σ_0 and I_1 by Equations 28a and b. Thus, taking the inverse square of Equation 32 and using Equation 28b we obtain

$$\sigma_0^4 = -b\sigma_0^2/(2n_2\bar{I}\omega_0/c) \tag{33}$$

We can write Equation 33 in the form

$$\sigma_0^2 = -b/(2n_2\bar{I}\omega_0/c) \tag{34}$$

Note that $b = \mathrm{d}^2k/\mathrm{d}\omega^2$, $\omega = \omega_0$ must be negative. As is well known in the theory of soliton† propagation in silica, this is the case at wavelengths larger than about 1.3 μm. Physically, Equation 34 says that, if the guide dispersion b is negative and large in magnitude, the pulse must have a long duration, unless the optical intensity is large. As b goes to 0, so does σ_0. But then higher order terms in the expansion in Equation 2 should be considered.

Equation 34 also says that the pulse amplitude area $\sigma_0\bar{I}^{1/2}$ is a constant. Thus, as shown for example by Linn *et al.*, [6] if the pulse energy is reduced by a factor F, σ_0 is increased by that same factor F. This increase may be much less than that due to dispersion without nonlinear effects (for numerical values, see, for example, Reference 6). If we stick to the assumed logarithmic nonlinearity, then σ_0, according to Equation 34 is independent of the pulse intensity. This is because, as the intensity gets smaller, the slope of the postulated $n(I)$ curve gets larger. For a truly linear dependence of n on I, the Gaussian solution discussed in this paper is only a very rough approximation of the actual $\mathrm{sech}(t)$ law.

6. Complex-ray tracing

In this section, we generalize somewhat the results of the previous section by considering nonstationary solutions. The propagation of Gaussian pulses in dispersive media with quadratic space–time inhomogeneities is fully described by a complex paraxial ray $q(z)$ as in the spatial case [2, 7].

To obtain the equation obeyed by $q(z)$, it suffices to set

$$U_g(z) = p(z)/q(z) \qquad p(z) = -b^{-1}\dot{q}(z) \tag{35}$$

in Equation 29. We obtain readily the linear equation

$$\ddot{q}(z) + (-k_0 b)\Omega^2(z)q(z) = 0 \tag{36}$$

Because Ω^2 is real, both the real and imaginary parts $q_r(z)$ and $q_i(z)$ of $q(z)$ must obey Equation 36. From the expression of the field

$$\psi \sim \exp(\mathrm{i}U_g t^2) = \exp[\mathrm{i}(-b^{-1}\dot{q}/q)t^2] \tag{37}$$

it is clear that the pulse duration is

$$\sigma(z) = |q(z)| = [q_r^2(z) + q_i^2(z)]^{1/2} \tag{38}$$

provided the invariant quantity

$$q_r\dot{q}_i - \dot{q}_r q_i \tag{39}$$

can be set equal to $-b$.

†Strictly speaking, solitary waves, since the stability of the solutions has not been established.

From Equation 38 it is easy to calculate the second derivative of $\sigma(z)$ with respect to z. Using Equation 39 we find

$$\ddot{\sigma} = b^2/\sigma^3 + k_0 b \Omega^2 \sigma \tag{40}$$

The first term on the right-hand side of Equation 40 expresses the effect of dispersion (diffraction in the analogous spatial problem). It makes the pulse spread since $\ddot{\sigma} > 0$. The second term on the contrary tends to make the pulse contract if $b\Omega^2 < 0$.

If the inhomogeneity is induced by the field through the logarithmic nonlinearity, we may use for Ω^2 the expression in Equation 28b with σ_0 replaced by $\sigma(z)$.

$$\Omega^2(z) = A/\sigma^2(z) \qquad A = 2(n_2/n_0)\bar{I} \tag{41}$$

Then the differential equation Equation 40 becomes

$$\ddot{\sigma} = b^2/\sigma^3 + k_0 b A/\sigma \tag{42}$$

The steady-state situation: $\ddot{\sigma} = 0$ brings us back to the soliton solution in Equation 34 [note that $k_0 = (\omega/c)n_0$], but Equation 42 is more general. Numerically, the direct integration of Equation 36 seems to be preferable.

7. Conclusion

We have presented a theory of Gaussian pulse propagation in linear and nonlinear media [8]* based on the concept of the complex-ray, or, equivalently, on the concept of complex-point eikonal. This apparently has not been done in detail before. For the sake of clarity, only simple cases were considered, and the results given may not be basically new. However, considerably more complicated systems, with anisotropy or losses, or devices such as those considered by Froehly *et al.* [8] can be treated with that formalism.

References

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*Earlier work on the logarithmic nonlinear equation was reported by Bialynicki-Birula. The complex-ray representation, however, was not used.