Analogy between optical rays and nonrelativistic particle trajectories: A comment

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The identification: medium refractive index n = nonrelativistic particle velocity v, in the analogy of geometrical optics with nonrelativistic particle mechanics is physically correct and useful, but can easily be misinterpreted. The conditions under which the analogy holds are outlined. The significance of Descartes's mechanical models is also discussed.

The identification:

$$\text{medium refractive index } n = \text{nonrelativistic particle velocity } v \quad (1)$$

in the analogy of geometrical optics with nonrelativistic particle mechanics has been clearly and correctly discussed by many authors (e.g., Refs. 1–4). Other authors, however, have suggested that this analogy may not be physically correct. It is the purpose of the present paper to clarify, in perhaps more detail than has been done elsewhere, the significance and limitations of that analogy. The difficulties that are most often encountered seem to be the following. The relation in (1) appears to be dimensionally incorrect, since n is dimensionless while v has the dimension of the ratio of length and time. In reality, only *proportionality* of v and n is intended in (1). The dimensions are therefore unimportant. A second, more fundamental difficulty is that velocity is obviously a time-dependent concept. On the other hand, the concept of time should not enter in the analogy, a point that we shall emphasize later. The solution of that apparent contradiction is that v in (1) is only a short-hand notation for $[-2V(\mathbf{x})]^{1/2}$, where V denotes the scalar, time-independent potential to which the massive particle is submitted. With that understanding, the mechanical analogy for ray trajectories given in (1) is both physically correct and useful, not only for predicting ray trajectories, but also for solving problems involving forces in the steady state.

Historically, geometrical optics has been concerned with ray trajectories defined as curves in ordinary space, while classical mechanics treats in addition the position of a particle along its trajectory as a function of time. Thus, for classical mechanics, the full space-time Hamilton equations are essential, while, in many problems of optics, the Hamilton equations can be restricted to the three-dimensional space. We need emphasize that the analogy in (1) holds between monochromatic light rays in lossless isotropic media and trajectories in free space of monoenergetic nonrelativistic particles submitted to time-invariant scalar potentials. The motion of wave packets as a function of time is not relevant to such a comparison. This point is made clear by Gabor[2] who writes "Hamilton's principle is too general for the purpose of electron optics. It contains time, which is without interest if the fields are stationary .... First, we fix the energy constant, that is to say, we restrict the discussion to monochromatic light and monoenergetic electrons." Similarly, Luneburg,[1] in his discussion of the mechanical analogy, warns his readers (in p. 85): "the velocity of the electron is greater in a medium of greater n, and therefore is *not* to be identified with the velocity of light on the rays, according to Huygens's definition" (that is, for nondispersive refracting media). We know, from the Heisenberg uncertainty relations, that a particle whose energy has a very precise value cannot be localized on its trajectory. Therefore, for such a particle, times of flight and velocities cannot be defined. Similarly, times of flight are undefined for optical rays that are strictly monochromatic.[5]

To be more specific, let us write the equation obeyed by the trajectories in space, $\mathbf{x}(t)$, of nonrelativistic particles having zero total energy in a gravitational potential $V(\mathbf{x})$. We have

$$\frac{d^2\mathbf{x}(t)}{dt^2} = -\nabla V(\mathbf{x}), \quad (2a)$$

where

$$\frac{|d\mathbf{x}|}{dt} \equiv [-2V(\mathbf{x})]^{1/2}. \quad (2b)$$

$|d\mathbf{x}|$ represents the elementary length of the trajectory, and the parameter t has the dimension of time.

Light rays in isotropic time-invariant media with refractive index $n(\mathbf{x})$ (where n is obtained, for example, from the measurement of the angle of refraction on homogeneous samples of the medium considered) obey the equation

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \nabla\left(\frac{n^2(\mathbf{x})}{2}\right), \quad (3a)$$

where

$$\frac{|d\mathbf{x}|}{dt} \equiv n(\mathbf{x}) \quad (3b)$$

and the parameter t has the dimension of a length.

Comparison of (2) and (3) shows that the equations obeyed by nonrelativistic particle and light ray trajectories are the same when $n^2(\mathbf{x})/2$ in (3) is proportional to $-V(\mathbf{x})$ in (2). The proportionality constant can be set equal to unity, for simplicity, with the appropriate dimensions. Thus, the fact that (1) seems to lack the proper dimensions is unimportant as Kline and Kay have pointed out.[3] Note that, in Eqs. (2) and (3), t is defined as a specific function of the length along the trajectory. It is defined by (2b) for nonrelativistic particles, and by (3b) for light rays.

We now arrive at a critical point: it is true, *but not relevant to the comparison made,* that t in (2) can be given the significance of a time of flight. In other words, it is correct, but perhaps misleading, to define the right-hand side of

$$\left|\frac{d\mathbf{x}}{dt}\right| = v(\mathbf{x}) \quad (4)$$

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as the magnitude of the velocity of the nonrelativistic particle.

Equations (2) and (3) can be given alternative equivalent forms, when differentiation with respect to the t parameter is replaced by differentiation with respect to the ray length s, with $ds = |d\mathbf{x}|$. We have, respectively,

$$V(\mathbf{x})^{1/2}\frac{d}{ds}\left(V(\mathbf{x})^{1/2}\frac{d\mathbf{x}}{ds}\right) = \nabla\left(\frac{V(\mathbf{x})}{2}\right)$$

(nonrelativistic particle trajectory) (3c)

and

$$n(\mathbf{x})\frac{d}{ds}\left(n(\mathbf{x})\frac{d\mathbf{x}}{ds}\right) = \nabla\left(\frac{n^2(\mathbf{x})}{2}\right)$$

(light ray trajectory). (3d)

Still another form is obtained when $\mathbf{x}$ is differentiated with respect to a parameter that varies monotonically along the trajectory, but which, unlike the parameters t or s used earlier, is *not* a specific function of the arc length (see, for example, Ref. 4, p. 293). The mathematics is simplified when such a parameter is used, but the formulation becomes more abstract. It need not be discussed further here.

The physical significance of the equivalence set in (1) is most obvious from a wave optics point of view. What (1) says is that spatial trajectories in isotropic time-invariant media depend only on the wave number law $k(\mathbf{x})$. Furthermore, the trajectories are unaffected if $k(\mathbf{x})$ is multiplied by an arbitrary constant. In optics, $k(\mathbf{x})$ is, by definition, proportional to the refractive index $n(\mathbf{x})$ of the medium, the frequency being kept fixed. For nonrelativistic particles submitted to a scalar potential $V(\mathbf{x})$, $k(\mathbf{x})$ is proportional to $v(\mathbf{x}) \equiv [-2V(\mathbf{x})]^{1/2}$, the total energy being kept equal to zero, according to the de Broglie relation. Thus, proportionality between the wave numbers at any point in space leads to relation (1). Therefore, if the refractive index is understood, as usual, as the ratio of the velocity of light in free space to the phase velocity in the medium at the point considered (and at some fixed frequency, or energy) the refractive index of a nonrelativistic particle with velocity v is correctly defined as a quantity proportional, or equal, to v. In short, $n = v$. Any other relation that we may think of (such as $n = 1/v$) would be incorrect for the problem considered. The conclusion that $n = v$ by no means implies that, inversely, the velocity of a light pulse in a medium with refractive index n has a velocity proportional to n. The reason why the argument cannot be taken the other way around is that a medium, in general, does not have the same dispersion as free space. Knowledge of the refractive index (n) of an arbitrary medium does not entail the knowledge of its dispersion ($dn/d\omega$), in contradistinction to the case of a particle in free space.

There is one more point that may cause difficulties in the analogy presently discussed: it is well known that the wave vector $\mathbf{k}$ associated with a charged particle is defined only within the gradient of an arbitrary function of $\mathbf{x}$. However, as one easily proves, ray trajectories are unaffected by this gradient. The same holds true for optical rays: the displacement at each point in space of the surface of wave normals, $H(k_x, k_y, k_z) = 0$ by a vector $\nabla f(\mathbf{x})$, where $f(\mathbf{x})$ is arbitrary, does not affect the optical ray trajectories. The medium, however, appears to be anisotropic after this transformation, and the Hamilton ray equations must be used in their general form.

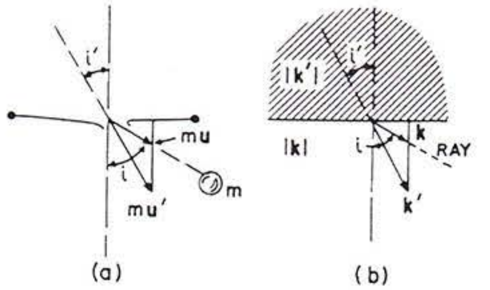

Fig. 1. Illustration of Descartes's mechanical analogy of the refraction of light rays. (a) A ball moving in free space traverses a membrane that reduces the magnitude of its momentum by a factor independent of the incidence angle. The tangential component of the momentum $m\mathbf{u}$ is invariant because the membrane does not exert any force on the ball in that direction. (b) A light ray is refracted away from the normal by going into a less dense medium. The tangential component of the vector $\mathbf{k}$, whose magnitude is independent of the incidence angle, is invariant. $\sin(i)/\sin(i')$ is a constant in both cases. The Descartes–Snell law of refraction follows quite generally from isotropy and translational invariance of the medium.

There are of course mechanical systems, different from the one considered above, to which the analogy in (1) does not apply. For example, if a charged particle has a varying mass because of radioactive decay, or a varying effective mass because of interaction with the atoms of a nonhomogeneous crystal or motion in a fluid, its trajectory cannot be compared to that of light rays on the basis of (1). The analogy that we have discussed is restricted to nonrelativistic particles of constant mass in free space. This still leaves us with the broad and important field of electron optics.[6]

Let us add one point of historical interest. It is generally agreed that the analogy in (1) was first proposed by the 17th century philosopher René Descartes. In his *Dioptrique* (1637), Descartes[7] points out an analogy between the trajectory of a light ray refracted at the boundary between two media, such as air and water, and the spatial trajectory of a ball traversing a membrane that reduces its velocity, in magnitude, by a factor independent of the angle of incidence. It is understood, in the Descartes analogy, that the ball, unlike the ray, is moving all the time in air,[8] and that the action of air, as well as that of gravity, can be neglected. This analogy is in agreement with modern concepts in physics provided that the quantity that Descartes calls the "determination" of the ball be understood as the ball "canonical momentum," which is proportional to the wave vector in the special case of lossless media[9] (see Fig. 1). As far as dynamical (force) concepts are concerned, Descartes has shown remarkable foresight in picturing light, in the steady state, as the transmission of some kind of pressure. Recent experiments[10] have shown that the pressure exerted by light on absorbers located in inviscid fluids (radiation pressure) is indeed equal to the canonical momentum of the ray. This result, in fact, holds for any linear wave.[11] It holds true, in particular, for the de Broglie matter waves associated with nonrelativistic particles.

In the above discussion, time has essentially been ignored. In many systems, however, the propagation of light pulses is of paramount importance. Readers interested in the propagation of pulses of electromagnetic radiation are referred to a beautiful paper by Weinberg,[12] based on the Hamilton equations in space–time. The condition under

which the Fermat principle can be considered a principle of minimum time, which is discussed in that paper, provides further clarification of the analogy in (1).

The existence of general methods to describe the motion in space–time of optical pulses obviously does not invalidate the analogy in (1), aimed at solving a simpler problem. It is desirable, and of course physically correct, to reduce the number of independent variables from four (space–time) to three (space), wherever possible. This can be done when the medium is stationary and the light is monochromatic. In some systems (e.g., in fiber optics) one is concerned with the propagation of optical pulses in media that are not only stationary, but also uniform along a spatial (say z) direction. In that case, the axial component, k_z, of the four-wave vector, as well as its time component, ω, are constants of motion. In some special cases (e.g., cold plasmas, thick dielectric slabs. . .), the axial group velocity, as well as k_z and ω, is a constant of motion, as is the case in similar problems of nonrelativistic mechanics.[9]

What makes the analogy in (1) so valuable is that the quantities introduced, namely n and v, have a simple and direct physical significance. This is not always the case for the more general quantities (e.g., canonical momenta) introduced by Hamilton. Differences in motivation and language between specialists in optics and in physics has unfortunately occasionally created a confusion concerning the significance of the mechanical analogy of light. It is hoped that the present article will help clarify the nature of the problem that the equivalence refractive index = velocity is intended to solve, and the limitations of that analogy.

ACKNOWLEDGMENT

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[1] R. K. Luneburg, *The Mathematical Theory of Light* (Berkeley U. P., Berkeley, CA, 1964).

[2] M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1970), Appendix II. The writing of this appendix is credited by the authors to D. Gabor.

[3] M. Kline and I. M. Kay, *Electromagnetic Theory and Geometrical Optics* (Wiley-Interscience, New York, 1965).

[4] J. A. Arnaud, in *Progress in Optics, Vol. 11*, edited by E. Wolf (North-Holland, Amsterdam, 1973).

[5] Times of flight should not be confused with transit times of crests (or nodes) of time-harmonic waves. The latter are closely related to the so-called "optical lengths." They are unrelated to the dispersion of the material.

[6] It is easy to generalize (1) to relativistic particles submitted to vector, as well as scalar, potentials, that is, to magnetic as well as electric fields [see P. Grivet, *Electron Optics* (Pergamon, Oxford, 1965)]. We have omitted discussion of this generalization here for the sake of clarity.

[7] R. Descartes, *Dioptrique* (1637); see *Oeuvres et Lettres, Collection: La Pléiade* (Gallimard, Paris, 1963).

[8] To avoid confusion, let us recall that Descartes gives subsequently in his *Dioptrique* a second mechanical analogy, quite independent of the first, where the ball moves from air into water. As in the previous model, Descartes assumes that the ball velocity is reduced in magnitude by a constant factor at the interface, but that, otherwise, water does not disturb the ball motion. In modern terms, this means that the ball is so dense that its apparent mass in water almost equals its actual mass, but that it loses a significant fraction of its energy as it enters water. These assumptions are qualitatively consistent with the "total reflection" experiment with cannon balls reported by Descartes. If the opposite assumptions were made—that the loss of energy at the air–water interface is negligible, but that the change of apparent mass of the ball is significant—an opposite direction of refraction would obtain. Thus, the true physical situation for this second model is rather complicated.

[9] J. A. Arnaud, *Beam and Fiber Optics* (Academic, New York, 1976).

[10] R. V. Jones and J. C. S. Richard, Proc. R. Soc. Lond. A **221**, 480 (1954).

[11] P. A. Sturrock, Phys. Rev. **121**, 18 (1961).

[12] S. Weinberg, Phys. Rev. **126**, 1899 (1962).